Heuristic Description of Perpendicular Diffusion of Energetic Particles in Astrophysical Plasmas

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Abstract

A heuristic approach for collisionless perpendicular diffusion of energetic particles is presented. Analytic forms for the corresponding diffusion coefficient are derived. The heuristic approach presented here explains the parameter a^2 used in previous theories in order to achieve agreement with simulations and its relation to collisionless Rechester & Rosenbluth diffusion. The obtained results are highly relevant for applications because previously used formulas are altered significantly in certain situations.

Key words: solar wind

1. Introduction

A problem of the utmost importance is the interaction between electrically charged particles and magnetized plasmas. It plays a significant role in a variety of physical systems ranging from fusion devices, over the solar wind, to the shock fronts of supernova explosions. In all of those scenarios energetic particles experience scattering due to complicated interactions with turbulent magnetic fields. Some early work was done based on perturbation theory also known as quasi-linear theory (see, e.g., Jokipii 1966), but in general this approach fails. Some more heuristic arguments but also systematic theories have been developed focusing on electron heat transport in fusion plasmas where collisions are assumed to play a significant role (see, e.g., Rechester & Rosenbluth 1978; Kadomtsev & Pogutse 1979; Krommes et al. 1983). In space plasmas such as the solar wind or the interstellar medium, on the other hand, collisions are absent and, thus, it was concluded that the aforementioned approaches are not applicable. The assumption of exponential field line separation was also questioned (see, e.g., Matthaeus et al. 2003). In the context of astrophysical plasmas, however, one still finds perpendicular diffusion in most cases as shown via test-particle simulations (see, e.g., Giacalone & Jokipii 1999; Qin et al. 2002), but it remained unclear what the mechanisms behind this type of transport are. Progress has been achieved due to the development of the nonlinear guiding center theory (Matthaeus et al. 2003), the unified nonlinear transport (UNLT) theory of Shalchi (2010), as well as its time-dependent generalization (Shalchi 2017). Within *diffusive UNLT theory* the perpendicular diffusion coefficient is given by

$$\kappa_\perp = \frac{a^2 v^2}{3 B_0^2} \int d^3k \, \frac{P_{xx}(\boldsymbol{k})}{F(k_\parallel, k_\perp) + (4/3)\kappa_\perp k_\perp^2 + v/\lambda_\parallel}, \quad (1)$$

where $F(k_\parallel, k_\perp) = (vk_\parallel)^2/(3\kappa_\perp k_\perp^2)$. The solution of this equation depends on the spectral tensor P_{nm} describing the magnetic fluctuations, the parallel mean free path $\lambda_\parallel = 3\kappa_\parallel/v$, the particle speed v, and the mean field B_0. Asymptotic solutions of Equation (1) and the importance of the Kubo number (Kubo 1963) defined via $K = (\ell_\parallel \delta B_x)/(\ell_\perp B_0)$, depending on the parallel and perpendicular bendover scales $\ell_\parallel$ and $\ell_\perp$ as well as the turbulent magnetic field δB_x, have been discussed in Shalchi (2015). Equation (1) shows good agreement with

most test-particle simulations, in particular with those performed for three-dimensional turbulence with small and intermediate Kubo numbers. Furthermore, Equation (1) contains quasi-linear theory as well as the nonlinear theory of field line random walk (FLRW) developed by Matthaeus et al. (1995). In Shalchi (2017) *time-dependent UNLT theory* has been derived, which is represented by

$$\frac{d^2}{dt^2}\langle(\Delta x)^2\rangle = \frac{2a^2}{B_0^2} \int d^3k \, P_{xx}(\boldsymbol{k}, t)\xi(k_\parallel, t)e^{-\frac{1}{2}\langle(\Delta x)^2\rangle k_\perp^2}, \quad (2)$$

with the parallel correlation function given by $\xi(k_\parallel, t) = (v^2/3)(\omega_+ e^{\omega_+ t} - \omega_- e^{\omega_- t})/(\omega_+ - \omega_-)$ where $\omega_\pm = -v/(2\lambda_\parallel) \pm [v^2/(2\lambda_\parallel)^2 - v^2 k_\parallel^2/3]^{1/2}$. Equation (1) can be derived from Equation (2) by employing a diffusion approximation. Furthermore, the theory explains why diffusion is restored and this is entirely due to transverse complexity becoming important. Due to the exponential factor in Equation (2), this means that diffusion is obtained if $\langle(\Delta x)^2\rangle \geqslant 2\ell_\perp^2$. However, there are at least two remaining problems in the theory of perpendicular diffusion. First, there is a discrepancy between theory and simulations in the large Kubo number regime that was previously balanced out by using the factor a^2 (see Equations (1) and (2)) and by setting $a^2 = 1/3$ (see Matthaeus et al. 2003). Furthermore, the question remains what the physics behind collisionless perpendicular diffusion is. This Letter provides an answer to both questions.

2. The Three Rules of Perpendicular Diffusion

We now formulate rules allowing us to derive formulas for the perpendicular diffusion coefficient without employing systematic theories. Those rules are:

1. Perpendicular transport is only controlled by three effects, namely, parallel transport, the random walk of magnetic field lines, as well as transverse complexity. The last of these three effects leads to the particles getting scattered away from the original magnetic field lines they were tied to.
2. We assume that the bendover scales $\ell_\parallel$ and $\ell_\perp$, the integral scales $L_\parallel$ and $L_\perp$, the ultra-scale L_U, as well as the

Table 1
The Eight Cases of Perpendicular Transport

Case	Parallel Motion	Field Lines	$\langle(\Delta x)^2\rangle \geqslant 2\ell_\perp^2$	Perpendicular Transport	Diffusion Coefficient	Described by UNLT Theory
1	Ballistic	Ballistic	No	Ballistic	$d_\perp(t) = \frac{v^2}{3}\frac{\delta B_x^2}{B_0^2}t$	Yes
2	Ballistic	Ballistic	Yes	Double-ballistic diffusion	$\kappa_\perp = \sqrt{\frac{2}{3}}v\ell_\perp\frac{\delta B_x}{B_0}$	Yes
3	Ballistic	Diffusive	No	FLRW Limit	$\kappa_\perp = \frac{v}{2}\kappa_{\mathrm{FL}}$	Yes
4	Ballistic	Diffusive	Yes	FLRW Limit	$\kappa_\perp = \frac{v}{2}\kappa_{\mathrm{FL}}$	Yes
5	Diffusive	Ballistic	No	Fluid Limit	$\kappa_\perp = \frac{\delta B_x^2}{B_0^2}\kappa_\parallel$	Yes
6	Diffusive	Ballistic	Yes	Fluid Limit	$\kappa_\perp = \frac{\delta B_x^2}{B_0^2}\kappa_\parallel$	Yes
7	Diffusive	Diffusive	No	Compound Subdiffusion	$d_\perp(t) = \kappa_{\mathrm{FL}}\sqrt{\frac{\kappa_\parallel}{2t}}$	Only for small Kubo numbers
8	Diffusive	Diffusive	Yes	CLRR Limit	$\kappa_\perp = \left(\frac{\kappa_{\mathrm{FL}}}{\ell_\perp}\right)^2\kappa_\parallel$	Only for small Kubo numbers

Note. The results are compared with limits contained in time-dependent UNLT theory represented by Equation (2).

Kolmogorov scale L_K are finite and nonzero. Furthermore, the parallel motion is assumed to be ballistic at early times and thereafter turns into a diffusive motion described by the parallel diffusion coefficient $\kappa_\parallel$. The FLRW is initially ballistic and becomes diffusive for larger distances. In this case it is described by the field line diffusion coefficient κ_{FL} which depends on some of the aforementioned scales.

3. In order to obtain normal diffusion, the particles need to leave the original magnetic field lines they followed. This happens as soon as transverse complexity becomes significant corresponding to[1]

$$\langle(\Delta x)^2\rangle \geqslant 2\ell_\perp^2. \tag{3}$$

What the perpendicular diffusion coefficient is depends solely on the state of parallel and field line transport at the time particles start to satisfy condition (3).

3. The Perpendicular Diffusion Coefficient

In the following we construct the perpendicular diffusion coefficient $\kappa_\perp$ based on the three rules formulated above. We shall derive eight cases that are summarized in Table 1. As demonstrated, there are four different routes to perpendicular diffusion as listed in Table 2.

3.1. The Field Line Random Walk Limit

First, we assume that the random walk of magnetic field lines is diffusive in the scenario of interest[2]

$$\langle(\Delta x)^2\rangle = 2\kappa_{\mathrm{FL}}|z|. \tag{4}$$

If we assume that there are no collisions and no pitch-angle scattering, we can set $z = v\mu t$ where we used the pitch-angle cosine μ. Combining this with Equation (4) and averaging over μ yields $\langle(\Delta x)^2\rangle = v\kappa_{\mathrm{FL}}t$, and thus

$$\kappa_\perp = \frac{v}{2}\kappa_{\mathrm{FL}} \tag{5}$$

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[1] It is assumed here that $\ell_\perp$ is the scale at which transverse complexity becomes significant. In principle, this could be a different scale such as the integral scale $L_\perp$. The use of the bendover scale, however, is motivated by time-dependent UNLT theory (see Section 1).

[2] Note that the diffusion coefficient κ_{FL} has length units.

Table 2
The Four Routes to Perpendicular Diffusion

Route	Final State	Diffusion Coefficient
$1 \to 2$	Double-ballistic diffusion	$\kappa_\perp = \sqrt{\frac{2}{3}}v\ell_\perp\frac{\delta B_x}{B_0}$
$1 \to 3 \to 4$	FLRW Limit	$\kappa_\perp = \frac{v}{2}\kappa_{\mathrm{FL}}$
$1 \to 5 \to 6$	Fluid Limit	$\kappa_\perp = \frac{\delta B_x^2}{B_0^2}\kappa_\parallel$
$1 \to 7 \to 8$	CLRR Limit	$\kappa_\perp = \left(\frac{\kappa_{\mathrm{FL}}}{\ell_\perp}\right)^2\kappa_\parallel$

Note. In the first three cases perpendicular transport starts as ballistic motion that then turns into a diffusive motion. In the fourth case the ballistic motion is followed by a subdiffusive regime and thereafter diffusion is restored.
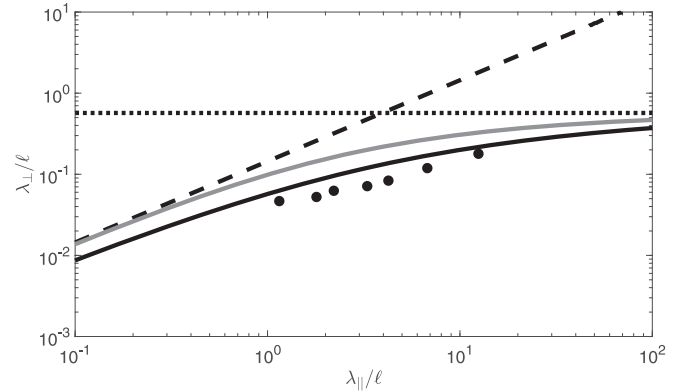

Figure 1. Results for a spectral tensor based on the critical balance condition of Goldreich & Sridhar (1995). For $\delta B/B_0 = 1$ the field line diffusion coefficient is in this case $\kappa_{\mathrm{FL}} = 0.38\ell$. Shown are the simulations (dots) of Jokipii (2011), the result of diffusive UNLT theory for $a^2 = 1$ (solid line), the CLRR limit (dashed line) as given by Equation (9), the FLRW limit (dotted line) as given by Equation (5), and the composite formula (gray line) as given by Equation (20).

corresponding to the FLRW limit. This limit is stable because if condition (3) is met, it does not alter the transport. This case is highly relevant in the limit of long parallel mean free paths corresponding to high particle energies (see Figures 1 and 2).
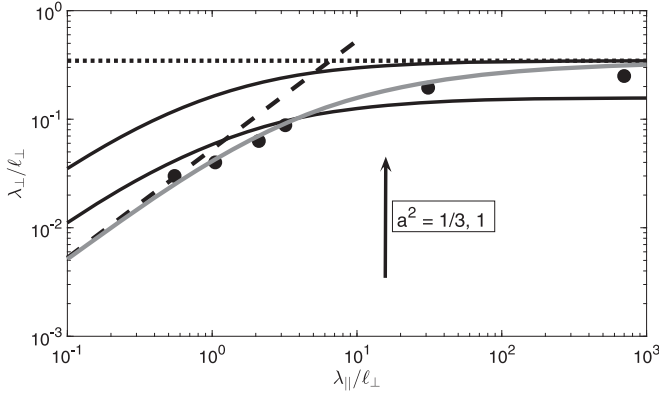
Figure 2. Results for a spectral tensor based on the NRMHD model of Ruffolo & Matthaeus (2013). For $\delta B/B_0 = 1$ the field line diffusion coefficient is in this case $\kappa_{\mathrm{FL}} = 0.23\ell_\perp$. Shown are the simulations (dots) of Shalchi & Hussein (2014), the results of diffusive UNLT theory for $a^2 = 1/3$ and $a^2 = 1$ (solid lines), the CLRR limit (dashed line) as given by Equation (9), the FLRW limit (dotted line) as given by Equation (5), and the composite formula (gray line) as given by Equation (20).

3.2. Compound Subdiffusion

If there is strong pitch-angle scattering the parallel motion is diffusive meaning that

$$\langle (\Delta z)^2 \rangle = 2\kappa_\parallel t. \tag{6}$$

Assuming that field lines are diffusive and particles follow field lines, we can combine Equations (4) and (6) to find

$$\langle (\Delta x)^2 \rangle \approx 2\kappa_{\mathrm{FL}} \sqrt{2\kappa_\parallel t}. \tag{7}$$

The running perpendicular diffusion coefficient is then

$$d_\perp(t) = \frac{1}{2}\frac{d}{dt}\langle (\Delta x)^2 \rangle \approx \kappa_{\mathrm{FL}} \sqrt{\frac{\kappa_\parallel}{2t}} \tag{8}$$

corresponding to subdiffusive transport. However, diffusion will be restored as soon as condition (3) is satisfied as discussed in the next paragraph. For slab turbulence, on the other hand, this condition is never satisfied due to $\ell_\perp = \infty$, and thus we find compound subdiffusion as the final state of perpendicular transport.

3.3. The Collisionless Rechester & Rosenbluth Regime

We now assume that diffusion is restored as soon as the particles scatter away from their original field lines. This happens as soon as the condition (3) is satisfied. We also assume that this happens after the particles travel the distance L_K in the parallel direction leading to $\kappa_\perp/\kappa_\parallel = \langle (\Delta x)^2 \rangle / \langle (\Delta z)^2 \rangle = \ell_\perp^2/L_K^2$. In order to eliminate L_K we use the field line diffusion coefficient $\kappa_{\mathrm{FL}} = \langle (\Delta x)^2 \rangle / (2|z|) = \ell_\perp^2/L_K$ yielding

$$\kappa_\perp \approx \left(\frac{\kappa_{\mathrm{FL}}}{\ell_\perp} \right)^2 \kappa_\parallel. \tag{9}$$

Alternatively, one can replace the scale $\ell_\perp$ therein so that

$$\kappa_\perp \approx \frac{\kappa_{\mathrm{FL}} \kappa_\parallel}{L_K} \tag{10}$$

in agreement with Equation (8) of Rechester & Rosenbluth (1978) as well as Equation (4) of Krommes et al. (1983). The

quantity L_K is either called the *Kolmogorov–Lyapunov length* or just the *Kolmogorov length* (see, e.g., Krommes et al. 1983). However, here L_K is not an *exponentiation length* but a characteristic distance along the mean field at which transverse complexity becomes significant. Furthermore, Equations (9) and (10) were obtained without assuming collisions, and thus we call this result the *collisionless Rechester & Rosenbluth (CLRR) limit*. One can also obtain this by using a slightly different derivation. We assume that we find compound subdiffusion until the particles satisfy condition (3), which happens at the diffusion time t_d so that Equations (7) and (8) become $2\ell_\perp^2 = 2\kappa_{\mathrm{FL}}\sqrt{2\kappa_\parallel t_d}$ as well as $\kappa_\perp = \kappa_{\mathrm{FL}}\sqrt{\kappa_\parallel/(2t_d)}$. Combining the latter two equations in order to eliminate t_d yields again Equation (9). In order to evaluate this further, we consider two subcases, namely, small and large values of the Kubo number, respectively. For small Kubo numbers the field line diffusion coefficient is given by the quasi-linear limit

$$\kappa_{\mathrm{FL}} = L_\parallel \frac{\delta B_x^2}{B_0^2}. \tag{11}$$

With this form Equation (9) becomes

$$\kappa_\perp \approx \left(\frac{L_\parallel}{\ell_\perp} \right)^2 \frac{\delta B_x^4}{B_0^4} \kappa_\parallel \propto \frac{\ell_\parallel^2}{\ell_\perp^2} \frac{\delta B_x^4}{B_0^4} \kappa_\parallel \tag{12}$$

in agreement with the scaling obtained from the diffusive UNLT theory in Shalchi (2015). Furthermore, we find

$$L_K \propto \frac{\ell_\perp^2}{\ell_\parallel} \frac{B_0^2}{\delta B_x^2}. \tag{13}$$

For large Kubo numbers, on the other hand, we have

$$\kappa_{\mathrm{FL}} = L_U \frac{\delta B_x}{B_0} \tag{14}$$

with the ultra-scale L_U. Equation (14) is either called the nonlinear or Bohmian limit and is similar compared to the field line diffusion coefficient obtained by Kadomtsev & Pogutse (1979). Therewith, Equation (9) becomes

$$\kappa_\perp \approx \left(\frac{L_U}{\ell_\perp} \right)^2 \frac{\delta B_x^2}{B_0^2} \kappa_\parallel, \tag{15}$$

and the Kolmogorov scale is $L_K = (\ell_\perp^2 B_0)/(L_U \delta B_x)$.

3.4. The Fluid Limit

Let us now assume that parallel transport is diffusive but magnetic field lines are still ballistic when the particles start to satisfy condition (3). Then we can derive $\langle (\Delta x)^2 \rangle = \langle (\Delta z)^2 \rangle \delta B_x^2/B_0^2 = 2\kappa_\parallel t \delta B_x^2/B_0^2$, and thus

$$\kappa_\perp = \frac{\delta B_x^2}{B_0^2} \kappa_\parallel, \tag{16}$$

which Krommes et al. (1983) called *the fluid limit*.

3.5. The Initial Free-streaming Regime

The simplest case is obtained for the early times when parallel and field line transport are ballistic. In this case

$$\langle(\Delta x)^2\rangle = \frac{\delta B_x^2}{B_0^2}\langle(\Delta z)^2\rangle = \frac{v^2}{3}\frac{\delta B_x^2}{B_0^2}t^2 \qquad (17)$$

so that

$$d_\perp(t) = \frac{v^2}{3}\frac{\delta B_x^2}{B_0^2}t \qquad (18)$$

corresponding to ballistic perpendicular transport. However, this is not a stable regime since we only find this type of transport before condition (3) is met.

3.6. Double-ballistic Diffusion

We now consider a scenario where the transport is still ballistic when the particles start to satisfy condition (3). Therefore, we use Equations (17) and (18) to derive $2\ell_\perp^2 = v^2\delta B_x^2 t_d^2/(3B_0^2)$ as well as $\kappa_\perp = v^2\delta B_x^2 t_d/(3B_0^2)$. Combining the latter two equations leads to

$$\kappa_\perp = \sqrt{\frac{2}{3}}v\ell_\perp\frac{\delta B_x}{B_0}. \qquad (19)$$

A similar result can be derived from Equation (2) by assuming a ballistic perpendicular motion.

3.7. Timescale Arguments

In order to determine which case is valid for which scenario, one needs to explore at which time a certain process takes place. In the parallel direction particles need to travel a parallel mean free path in order to get diffusive, and thus $t_\parallel = \lambda_\parallel/v = 3\kappa_\parallel/v^2$. In the following we focus on the case of short $\lambda_\parallel$. For small Kubo numbers the field lines become diffusive for $|z| \approx \ell_\parallel$ and the corresponding time is $t_{FL} \approx \ell_\parallel^2/(2\kappa_\parallel)$. Then, on the other hand, if we assume that condition (3) is satisfied while the field lines are still ballistic, we have $2\ell_\perp^2 = 2\kappa_\parallel t_{Fluid}\delta B_x^2/B_0^2$. For $t_\parallel < t_{Fluid} < t_{FL}$ the final state is the fluid limit because then we find that parallel transport becomes diffusive first and then we meet condition (3). If, on the other hand, $t_\parallel < t_{FL} < t_{Fluid}$ the field lines become diffusive before condition (3) is met. This means that we find compound subdiffusion first. At even later times condition (3) is eventually met and diffusion is restored. The corresponding diffusion coefficient is then the CLRR limit. Using the formulas for the times discussed above, this means that we find CLRR diffusion for $\lambda_\parallel^2 \ll \ell_\parallel^2 \ll \ell_\parallel L_K$ where the Kolmogorov length L_K is given by Equation (13). Thus, for $\lambda_\parallel \ll \ell_\parallel$ we either find the fluid limit or CLRR diffusion. If additionally $\ell_\parallel \gg L_K$ we find the fluid limit, but for $\ell_\parallel \ll L_K$ we get CLRR diffusion. It follows from Equation (13) that $L_K/\ell_\parallel \approx K^{-2} \gg 1$ meaning that for small Kubo numbers we always find CLRR diffusion. For large Kubo numbers similar considerations can be made.

3.8. A Composite Formula

A problem of the heuristic approach is that the obtained formulas are only valid in asymptotic limits. Since the two most important cases are CLRR and FLRW limits, we propose for the perpendicular mean free path defined via $\lambda_\perp = 3\kappa_\perp/v$

the formula

$$\frac{\lambda_\perp}{\ell_\perp} = \frac{9\ell_\perp}{16\lambda_\parallel}\left[\sqrt{1 + \frac{8\kappa_{FL}\lambda_\parallel}{3\ell_\perp^2}} - 1\right]^2. \qquad (20)$$

Equation (20) was chosen so that for $\lambda_\parallel \to 0$ we obtain Equation (9) and for $\lambda_\parallel \to \infty$ we get Equation (5). Note that Equation (20) does not contain the fluid limit given by Equation (16), and thus it has some limitations.

3.9. Further Comments

The results obtained here are sometimes not comparable to previous results. First of all there are cases such as slab or two-dimensional (2D) turbulence. In the former case condition (3) is never satisfied leading to compound subdiffusion as the final state. In the 2D case parallel transport is not diffusive (see, e.g., Arendt & Shalchi 2018) violating the second rule. In some work (see, e.g., Matthaeus et al. 2003; Shalchi et al. 2004) a flat spectrum at large scales was used for the 2D modes. For this type of spectrum the ultra-scale is not finite, also violating the second rule. In order to determine the form of κ_{FL}, we have used the Kubo number. However, in some turbulence models (see, e.g., Goldreich & Sridhar 1995) there is only one scale, and thus the Kubo number becomes $K = \delta B_x/B_0$, often called the *Alfvénic Mach number*. The arguments presented above are still valid.

4. Comparison between Theory and Simulations

As a first example we consider two-component turbulence with dominant 2D modes. For a well-behaving spectrum, Shalchi & Weinhorst (2009) have derived

$$L_U = \sqrt{\frac{s-1}{q-1}}\ell_\perp \qquad (21)$$

requiring $q > 1$ for the energy range spectral index and $1 < s < 2$ for the inertial range spectral index. With the parameter a^2 included, nonlinear theories provide in the limit of short parallel mean free paths and 2D turbulence (see, e.g., Shalchi et al. 2004 and Zank et al. 2004)

$$\kappa_\perp = a^2\frac{\delta B_x^2}{B_0^2}\kappa_\parallel. \qquad (22)$$

According to the heuristic approach we expect CLRR diffusion in the considered parameter regime. Comparing Equations (22) and (15) yields $a = L_U/\ell_\perp$ and using Equation (21) for the ultra-scale gives us $a^2 = (s-1)/(q-1)$. Previously it was often assumed that $s = 5/3$ and $q = 3$ (see, for instance, Arendt & Shalchi 2018) leading to $a^2 = 1/3$. Although it was already stated in Matthaeus et al. (2003) that $a^2 = 1/3$ is needed to achieve agreement between theory and simulations, in the current Letter we found for the first time an explanation for this value. It has to be noted that this result was obtained for a specific form of the spectrum. Alternative spectra and the associated turbulence scales have been discussed in Matthaeus et al. (2007). For some of those spectra one obtains an ultra-scale larger than the bendover scale. In such cases, however, one would expect that the diffusion coefficient is close to the fluid limit, and thus $a^2 \approx 1$.

Two further examples are shown in Figures 1 and 2, respectively. A spectral tensor based on the critical balance condition of Goldreich & Sridhar (1995) was used in the simulations of Sun & Jokipii (2011). Figure 1 compares diffusive UNLT theory and the heuristic approach presented in the current paper with those simulations. As shown, the FLRW and CLRR limits have to be understood as asymptotic limits. Figure 2 visualizes the comparison for a spectral tensor based on the *noisy reduced MHD* (*NRMHD*) *model* of Ruffolo & Matthaeus (2013) showing very good agreement.

The heuristic arguments presented in this Letter cannot substitute systematic theories due to the lack of accuracy in the general case. The remaining step is to further improve UNLT theory so that the factor a^2 is no longer needed. This should lead to a complete systematic theory for perpendicular transport in space plasmas.

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