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Solutions of Sequential Conformable Fractional Differential Equations around an Ordinary Point and Conformable Fractional Hermite Differential Equation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

In this work, we give the power series solutions around an ordinary point, in the case of variable coefficients, homogeneous sequential linear conformable fractional differential equations of order 2α. Further, we introduce the conformable fractional Hermite differential equations, conformable fractional Hermite polynomials and basic properties of these polynomials.

Keywords: Sequential conformable fractional differential equation; ordinary point; conformable fractional Hermite differential equation; conformable fractional Hermite polynomials.

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1. INTRODUCTION

The idea of fractional derivative was raised first by L'Hospital in 1695. Since then, related to the definition of fractional derivatives have been given many definitions. The most popular ones of these definitions are Grunwald-Letnikov, Riemann-Liouville and Caputo definitions. For Riemann-Liouville, Caputo and other definitions and the characteristics of these definitions, we refer to the reader to [1-3].

Although the fractional calculus was not striking for a long time, it became the most popular working area along with fractional differential equations after its powerfull applications showed up and there has been made a lot of studies related to its theory and physical applications [4-10].

Since the analytic methods fell behind for exact solutions of the most fractional differential equations, there has been a tendency towards numerical and approximate analytic solution methods [11-17].

Recently, Khalil et al. [18] give a new definition of fractional derivative and fractional integral in. This new definition benefit from a limit form as in usual derivatives. They also proved the product rule, the fractional Rolle theorem and mean value theorem. In [19], Abdeljawad improve this new theory. For instance, definitions of left and right conformable fractional derivatives and fractional integrals of higher order (i.e. of order $\alpha > 1$), Taylor power series representation and Laplace transform of few certain functions, fractional integration by parts formulas, chain rule and Gronwall inequality are provided by him. The definition is found attractive and a large number of studies has been applied in this field in a short time [20-22].

In this work, we analyze the existence of solutions around an ordinary point of conformable fractional differential equation of order 2α. Then, we give solution of Hermite fractional differential equation. For this solution, we obtain Hermite fractional polynomials with certain special initial conditions. Finally, we introduce the basic properties of Hermite fractional polynomials.

2. CONFORMABLE FRACTIONAL CALCULUS

Definition 2.1. [18] Given a function $f: [0, \infty) \to \mathbb{R}$. Then the conformable fractional derivative of f order α is defined by

$$f^{(\alpha)}(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}$$

for all $t > 0, \alpha \in (0,1]$.

Theorem 2.1. [18] Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $x > 0$. Then

(1) $\frac{d^\alpha}{dx^\alpha}(af + bg) = a\frac{d^\alpha f}{dx^\alpha} + b\frac{d^\alpha g}{dx^\alpha}$, for all $a, b \in \mathbb{R}$
(2) $\frac{d^\alpha}{dx^\alpha}(x^p) = px^{p-\alpha}$, for all $p \in \mathbb{R}$
(3) $\frac{d^\alpha}{dx^\alpha}(\lambda) = 0$, for all constant functions $f(x) = \lambda$
(4) $\frac{d^\alpha}{dx^\alpha}(fg) = f\frac{d^\alpha}{dx^\alpha}(g) + g\frac{d^\alpha}{dx^\alpha}(f)$
(5) $\frac{d^\alpha}{dx^\alpha}(f/g) = \frac{g\frac{d^\alpha}{dx^\alpha}(f) - f\frac{d^\alpha}{dx^\alpha}(g)}{g^2}$
(6) If, in addition, f is differentiable, then $\frac{d^\alpha}{dx^\alpha}\big(f(x)\big) = x^{1-\alpha}\frac{df}{dx}(x)$.

Theorem 2.2. [19] Assume f is infinitely α-differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of a point x_0. Then f has the fractional power series expansion:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(\alpha k)}(x_0)(x - x_0)^{\alpha k}}{\alpha^k k!},$$

$$x_0 < x < x_0 + R^{1/\alpha}, \qquad R > 0.$$

Here, $f^{(\alpha k)}(x_0)$ means the application of the fractional derivative k times.

3. CONFORMABLE SEQUENTIAL FRACTIONAL DIFFERENTIAL EQUATION AND SOLUTIONS AROUND AN ORDINARY POINT

The most general sequential linear homogeneous conformable fractional differential equation is

$$ {}^{n}T_\alpha y + a_{n-1}(x)\,{}^{n-1}T_\alpha y + \ldots + a_1(x)T_\alpha y + a_0(x)y = 0, \tag{1}$$

where ${}^{n}T_\alpha y = T_\alpha T_\alpha \ldots T_\alpha y$, n times.

Definition 3.1. Let $\alpha \in (0,1]$, $x_0 \in [a,b]$, $N(x_0)$ be a neighborhood of x_0 and $f(x)$ be a real function defined on $[a,b]$. In this case $f(x)$ is said to be α-analytic at x_0 if $f(x)$ can be expressed as a series of natural powers of $(x-x_0)^\alpha$ for all $x \in N(x_0)$. In other word, $f(x)$ can be expressed as following:

$$\sum_{k=0}^{\infty} c_k (x-x_0)^{k\alpha} \qquad (c_k \in R)$$

This series being definitely convergent for $|x-x_0| < \delta \quad (\delta > 0)$. δ is the radius of convergence of the series.

Definition 3.2. Let $\alpha \in (0,1]$, $x_0 \in [a,b]$ and the functions $a_k(x)$ be α-analytic at $x_0 \in [a,b]$ for $k = 0,1,2,\ldots,n-1$. In this case, the point $x_0 \in [a,b]$ is said to be an α-ordinary point of (1). If a point $x_0 \in [a,b]$ is not α-ordinary point, then it is said to be α singular.

Example 3.1. a) We consider following the conformable fractional differential equations:

$$T_\alpha y - x^\alpha y = 0,$$

$$^2T_\alpha y - 2x^\alpha y = 0,$$

$$x^{2\alpha}\ {}^2T_\alpha y - 2x^\alpha T_\alpha y + x^{2\alpha} y = 0$$

Any point $x = x_0 > 0$ is an ordinary point for the above equations.

b) Let be

$$(x-1)^\alpha T_\alpha y - y = 0,$$

$$(x-1)^{2\alpha}\ {}^2T_\alpha y - 2(x-1)^\alpha T_\alpha y + (x-1)^{2\alpha} y = 0.$$

For these equations, any point $x = x_0 > 1$ is an ordinary point.

Theorem 3.1. Let $\alpha \in (0,1]$ and $x_0 \in [a,b]$ be an α-ordinary point of the equation

$$T_\alpha T_\alpha y + p(x) T_\alpha y + q(x) y = 0. \tag{2}$$

Then, there exists a solution to the equation (2) as

$$y = \sum_{k=0}^{\infty} c_k (x-x_0)^{k\alpha} \tag{3}$$

for $x \in (x_0, x_0+\rho)$ with $\rho = \min\{\delta_1,\delta_2\}$ and initial conditions $c_0 = y(x_0)$, $\alpha c_1 = T_\alpha y(x_0)$.

Proof. Since x_0 is an α-ordinary point of (2), by definition 3.1 and 3.2 we can write

$$p(x) = \sum_{k=0}^{\infty} p_k (x-x_0)^{k\alpha} \qquad (x \in [x_0, x_0 + \delta 1; \delta 1>0 \tag{4}$$

and

$$q(x) = \sum_{k=0}^{\infty} q_k (x-x_0)^{k\alpha}$$

$$(x \in [x_0, x_0+\delta_2];\ \delta_2 > 0). \tag{5}$$

We seek a solution in form (3) of (2). Substituting (3) and its conformable fractional derivatives in (2), then we obtain

$$\sum_{k=0}^{\infty} \alpha^2 (k+2)(k+1) c_{k+2} (x-x_0)^{k\alpha} + \left(\sum_{k=0}^{\infty} p_k (x-x_0)^{k\alpha}\right)\left(\sum_{k=0}^{\infty} \alpha(k+1) c_{k+1} (x-x_0)^{k\alpha}\right) + \left(\sum_{k=0}^{\infty} q_k (x - x0k\alpha k=0\infty ckx-x0k\alpha=0 \tag{6}$$

We also can write

$$\left(\sum_{k=0}^{\infty} p_k (x-x_0)^{k\alpha}\right)\left(\sum_{k=0}^{\infty} \alpha(k+1) c_{k+1} (x-x_0)^{k\alpha}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \alpha(j+1) p_{k-j} c_{j+1}\right)(x-x_0)^{k\alpha} \tag{7}$$

and

$$\left(\sum_{k=0}^{\infty} q_k (x-x_0)^{k\alpha}\right)\left(\sum_{k=0}^{\infty} c_k (x-x_0)^{k\alpha}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} q_{k-j} c_j\right)(x-x_0)^{k\alpha}. \tag{8}$$

Hence, if we substitute (7) and (8) in (6), we obtain

$$\sum_{k=0}^{\infty}\left[\alpha^2(k+2)(k+1)c_{k+2}+\sum_{j=0}^{k}\alpha(j+1)p_{k-j}c_{j+1}+\sum_{j=0}^{k}q_{k-j}c_j\right](x-x_0)^{k\alpha}=0.$$

So, the coefficients c_k must satisfy

$$\alpha^2(k+2)(k+1)c_{k+2}=-\sum_{j=0}^{k}\left[\alpha(j+1)p_{k-j}c_{j+1}+q_{k-j}c_j\right]. \quad (9)$$

We show that if the coefficient c_k are defined by (9), for $k\geq 2$, then the series

$$y=\sum_{k=0}^{\infty}c_k(x-x_0)^{k\alpha}$$

is convergent for $|x-x_0|<\rho$. Let us fix r $(0<r<\rho)$. Since the series in (4) and (5) are convergent for $|x-x_0|=r$, there is a constant $M>0$ such that

$$\left|p_{k-j}\right|\leq\frac{Mr^{j\alpha}}{r^{k\alpha}} \qquad (k\in N_0;0\leq j\leq k) \quad (10)$$

and

$$\left|q_{k-j}\right|\leq\frac{Mr^{j\alpha}}{r^{k\alpha}} \qquad (k\in N_0;0\leq j\leq k). \quad (11)$$

Using (10) and (11) in (9), we obtain

$$\begin{aligned}&\alpha^2(k+2)(k+1)|c_{k+2}|\leq\frac{M}{r^{k\alpha}}\sum_{j=0}^{k}\left[\alpha(j+1)\left|c_{j+1}\right|+\left|c_j\right|\right]r^{j\alpha}\\&\leq\frac{M}{r^{k\alpha}}\sum_{j=0}^{k}\left[\alpha(j+1)\left|c_{j+1}\right|+\left|c_j\right|\right]r^{j\alpha}+M\left|c_{j+1}\right|r^{\alpha}.\end{aligned} \quad (12)$$

Now, we define

$$C_0=|c_0|,C_1=|c_1|$$

and C_k by

$$\alpha^2(k+2)(k+1)C_{k+2}=\frac{M}{r^{k\alpha}}\sum_{j=0}^{k}\left[\alpha(j+1)C_{j+1}+C_j\right]r^{j\alpha}+MC_{k+1}r^{\alpha} \quad (13)$$

for $k\geq 2$.

We can see that an induction yields

$$|c_k|\leq C_k, \qquad C_k\geq 0, \qquad (k=0,1,2,\dots)$$

Now, we analyze for what x the series

$$\sum_{k=0}^{\infty}C_k(x-x_0)^{k\alpha} \quad (14)$$

is convergent.

Using (13), we obtain

$$\alpha^2(k)(k+1)C_{k+1}=\frac{M}{r^{(k-1)\alpha}}\sum_{j=0}^{k-1}\left[\alpha(j+1)C_{j+1}+C_j\right]r^{j\alpha}+MC_kr^{\alpha} \quad (15)$$

$$\alpha^2(k)(k-1)C_k = \frac{M}{r^{(k-2)\alpha}}\sum_{j=0}^{k-2}\left[\alpha(j+1)C_{j+1}+C_j\right]r^{j\alpha} + MC_{k-1}r^{\alpha}. \tag{16}$$

From (15) and (16), we find

$$r^{\alpha}\alpha^2(k)(k+1)C_{k+1} = \alpha^2(k)(k-1)C_k + \alpha Mkr^{\alpha}C_k + MC_kr^{2\alpha}.$$

Hence,

$$\frac{C_{k+1}}{C_k} = \frac{\alpha^2(k)(k-1)+\alpha Mkr^{\alpha}+Mr^{2\alpha}}{r^{\alpha}\alpha^2(k)(k+1)}$$

is obtained. By the help of the ratio test, we have that

$$\lim_{k\to\infty}\left|\frac{C_{k+1}(x-x_0)^{(k+1)\alpha}}{C_k(x-x_0)^{k\alpha}}\right| = \left(\frac{|x-x_0|}{r}\right)^{\alpha} < 1$$

Thus, the series (14) converges for $|x-x_0|<r$. This implies that the series (3) converges for $|x-x_0|<r$. Since r was any number satisfying $0<r<\rho$, the series (3) converges for $|x-x_0|<\rho$.

Example 3.2. Find the general solution to the equation

$$^{2}T_{\alpha}y - x^{\alpha}T_{\alpha}y - y = 0\,. \tag{17}$$

We seek a solution in the form (3). Substituting (3) and conformable fractional derivatives of (3) in (17), we have

$$c_2 = \frac{1}{2\alpha^2}c_0$$

and

$$c_{k+2} = \frac{k\alpha+1}{\alpha^2(k+1)(k+2)}c_k \quad k = 1,2,\ldots$$

Hence,

$$\begin{array}{cc}
c_2 = \frac{1}{2\alpha^2}c_0 & c_3 = \frac{2\Gamma\left(\frac{1-\alpha}{2\alpha}+2\right)}{\alpha.\Gamma\left(\frac{1}{2\alpha}+1\right)\Gamma(4)}c_1 \\
c_4 = \frac{\Gamma\left(\frac{1}{2\alpha}+2\right)}{2^{-1}\alpha^3\Gamma\left(\frac{1}{2\alpha}+1\right)\Gamma(5)}c_0 & c_5 = \frac{2^2\Gamma\left(\frac{1-\alpha}{2\alpha}+3\right)}{\alpha^2\Gamma\left(\frac{1-\alpha}{2\alpha}+1\right)\Gamma(6)}c_1 \\
c_6 = \frac{\Gamma\left(\frac{1}{2\alpha}+3\right)}{2^{-2}\alpha^4\Gamma\left(\frac{1}{2\alpha}+1\right)\Gamma(7)}c_0 & c_7 = \frac{2^3\Gamma\left(\frac{1-\alpha}{2\alpha}+4\right)}{\alpha^3\Gamma\left(\frac{1-\alpha}{2\alpha}+1\right)\Gamma(8)}c_1 \\
 & \vdots \\
c_{2k} = \frac{\Gamma\left(\frac{1}{2\alpha}+k\right)}{2^{1-k}\alpha^{k+1}\Gamma\left(\frac{1}{2\alpha}+1\right)\Gamma(2k+1)}c_0 & c_{2k+1} = \frac{2^k\Gamma\left(\frac{1-\alpha}{2\alpha}+k+1\right)}{\alpha^k\Gamma\left(\frac{1-\alpha}{2\alpha}+1\right)\Gamma(2k+2)}c_1
\end{array}$$

is obtained. The general solution of (17) is founded as

$$y(x) = c_0\sum_{k=0}^{\infty}\left[\frac{\Gamma\left(\frac{1}{2\alpha}+k\right)}{2^{1-k}\alpha^{k+1}\Gamma\left(\frac{1}{2\alpha}+1\right)\Gamma(2k+1)}\right]x^{2k\alpha} + c_1\sum_{k=0}^{\infty}\left[\frac{2^k\Gamma\left(\frac{1-\alpha}{2\alpha}+k+1\right)}{\alpha^k\Gamma\left(\frac{1-\alpha}{2\alpha}+1\right)\Gamma(2k+2)}\right]x^{(2k+1)\alpha}$$

4. CONFORMABLE SEQUENTIAL FRACTIONAL HERMITE DIFFERENTIAL EQUATION AND CONFORMABLE FRACTIONAL HERMITE POLYNOMIALS

Consider the conformable fractional Hermite differential equation

$$^{2}T_{\alpha}y - 2\alpha x^{\alpha}T_{\alpha}y + 2\alpha^{2}my = 0 \tag{18}$$

where $\alpha \in (0,1]$, m is a real number. If $\alpha = 1$, then equation (18) becomes the classical Hermite differential equation. Let m be a nonnegative integer. $x = 0$ is an ordinary point of (18). Now we seek a solution as in (3) of (18). Substituting (3) and its conformable fractional derivatives in (18), we have

$$c_2 = -mc_0$$

and

$$c_{k+2} = \frac{2(k-m)}{(k+1)(k+2)}c_k \quad k = 1,2,\ldots$$

Hence,

$$\begin{array}{ll} c_2 = (-1)\frac{2m}{2!}c_0 & c_3 = (-1)\frac{2(m-1)}{3!}c_1 \\ c_4 = (-1)^2\frac{2^2m(m-2)}{4!}c_0 & c_5 = (-1)^2\frac{2^2(m-1)(m-3)}{5!}c_1 \\ c_6 = (-1)^3\frac{2^3m(m-2)(m-4)}{6!}c_0 & c_7 = (-1)^3\frac{2^3(m-1)(m-3)(m-5)}{7!}c_1 \\ & \vdots \\ c_{2k} = (-1)^k\frac{2^k m(m-2)\ldots(m-2k+2)}{(2n)!}c_0 & c_{2k+1} = (-1)^k\frac{2^k(m-1)(m-3)\ldots(m-2k+1)}{(2n+1)!}c_1 \end{array}$$

is obtained. The general solution of (18) is found as

$$y(x) = c_0 + c_0\sum_{k=1}^{\infty}\left[(-1)^k\frac{2^k m(m-2)\ldots(m-2k+2)}{(2n)!}\right]x^{2k\alpha} + c_1x^{\alpha} + c_1\sum_{k=1}^{\infty}\left[(-1)^k\frac{2^k(m-1)(m-3)\ldots(m-2k+1)}{(2k+1)!}\right]x^{(2k+1)\alpha}$$

Now, we pick initial conditions the following as

$$y(0) = c_0 = (-2)^{\frac{m}{2}}(m-1)!!$$

$$T_{\alpha}y(0) = \alpha c_1 = 0 \rightarrow c_1 = 0$$

where

$$(m-1)!! = \begin{cases}(m-1)(m-3)\ldots 3.1 & (m-1)\ odd \\ (m-1)(m-3)\ldots 4.2 & (m-1)\ even\end{cases}$$

If $c_1 = 0$, then all $c_k = 0$ when k is odd. For these initial conditions, the solution is

$$y(x) = (-2)^{\frac{m}{2}}(m-1)!!\left[1 + \sum_{k=1}^{\infty}\left[(-1)^k\frac{2^k m(m-2)\ldots(m-2k+2)}{(2k)!}\right]x^{2k\alpha}\right]. \tag{19}$$

Specially, for $m = 6$, the solution is

$$y(x) = 64x^{6\alpha} - 480x^{4\alpha} + 720x^{2\alpha} - 120.$$

This is the 6th order conformable fractional Hermite polynomial. That is

$$H_6^\alpha(x) = 64x^{6\alpha} - 480x^{4\alpha} + 720x^{2\alpha} - 120.$$

To get the odd order Hermite polynomials, we specify the initial conditions

$$y(0) = c_0 = 0,$$

$$T_\alpha y(0) = \alpha c_1 = -\alpha(-2)^{\frac{m+1}{2}}(m)!! \to c_1 = -(-2)^{\frac{m+1}{2}}(m)!!.$$

If $c_0 = 0$, then all $c_k = 0$ when k is even. For these initial conditions, the solution is

$$y(x) = -(-2)^{\frac{m+1}{2}}(m)!!\left[x^\alpha + \sum_{k=1}^{\infty}\left[(-1)^k \frac{2^k(m-1)(m-3)\dots(m-2k+1)}{(2k+1)!}\right]x^{(2k+1)\alpha}\right] \tag{20}$$

Specially, for $m = 5$, the solution is

$$y(x) = 32x^{5\alpha} - 160x^{3\alpha} + 120x^\alpha.$$

This is the 5th order conformable fractional Hermite polynomial. That is

$$H_5^\alpha(x) = 32x^{5\alpha} - 160x^{3\alpha} + 120x^\alpha.$$

Properties of classical hermite polynomials [23] can be generalized to fractional hermite polynomials as following:

Property 4.1.

- **(I)** $H_m^\alpha(x) = H_m(x^\alpha)$
- **(II)** $T_\alpha\left(H_m^\alpha(x)\right) = 2m\alpha H_{m-1}^\alpha(x)$
- **(III)** ${}^mT_\alpha\left(H_m^\alpha(x)\right) = 2^m m!\,\alpha^m$
- **(IV)** $H_{m+1}^\alpha(x) = 2x^\alpha H_m^\alpha(x) - 2mH_{m-1}^\alpha(x)$
- **(V)** $H_{m+1}^\alpha(x) = 2x^\alpha H_m^\alpha(x) - \alpha^{-1}T_\alpha\left(H_m^\alpha(x)\right)$
- **(VI)** $H_m^\alpha(x) = (-\alpha^{-1})^m e^{x^{2\alpha}}\,{}^mT_\alpha\left(e^{-x^{2\alpha}}\right)$
- **(VII)** $\int_{-\infty}^{\infty} H_m^\alpha(x)\, H_n^\alpha(x) e^{-x^{2\alpha}} d_\alpha(x) = 0 \quad m \neq n \; and \; \alpha = \frac{1}{2j+1} \quad j \in N$
- **(VIII)** $\int_{-\infty}^{\infty} H_m^\alpha(x)\, H_n^\alpha(x) e^{-x^{2\alpha}} d_\alpha(x) = \frac{1}{\alpha}2^n n!\sqrt{\pi} \quad m = n \; and \; \alpha = \frac{1}{2j+1} \quad j \in N$

Proof.

- **(I)** Proof is obvious.
- **(II)** Let m be even. Then, $H_m^\alpha(x)$ is found by the help of (19). Applying conformable derivative to (19), we get

$$T_\alpha\left(H_m^\alpha(x)\right) = 2m\alpha\left[-(-2)^{\frac{m}{2}}(m-1)!!\left[x^\alpha + \sum_{k=1}^{\infty}\left((-1)^k \frac{2^k(m-2)\dots(m-2k)}{(2k+1)!}\right)x^{(2k+1)\alpha}\right]\right].$$

For $m-1$ is odd, we can write

$$T_\alpha\left(H_m^\alpha(x)\right) = 2m\alpha H_{m-1}^\alpha(x).$$

Conversely, let m be odd. Then $H_m^\alpha(x)$ is found by the help of (20). The conformable derivative of order α is

$$T_\alpha\left(H_m^\alpha(x)\right) = 2m\alpha\left[(-2)^{\frac{m-1}{2}}(m-2)!!\left[1+\sum_{k=1}^{\infty}\left((-1)^k\frac{2^k(m-1)(m-3)\dots(m-2k+1)}{(2k)!}\right)x^{(2k)\alpha}\right]\right].$$

For $m-1$ is even, similarly, we can write

$$T_\alpha\left(H_m^\alpha(x)\right) = 2m\alpha H_{m-1}^\alpha(x). \tag{21}$$

(III) We use induction to prove. For $m=1$, property is provided. That is

$$T_\alpha\left(H_1^\alpha(x)\right) = 2\alpha.$$

Assume that the property is true for $m=n$. That is

$${}^{n}T_\alpha\left(H_n^\alpha(x)\right) = 2^n n!\,\alpha^n. \tag{22}$$

From (II), we have

$$T_\alpha\left(H_{n+1}^\alpha(x)\right) = 2(n+1)\alpha H_n^\alpha(x). \tag{23}$$

Applying conformable fractional derivative of order α, n times, we get

$${}^{n+1}T_\alpha\left(H_{n+1}^\alpha(x)\right) = 2(n+1)\alpha\, {}^{n}T_\alpha\left(H_n^\alpha(x)\right). \tag{24}$$

Substituting (22) in (24), we obtain

$${}^{n+1}T_\alpha\left(H_{n+1}^\alpha(x)\right) = 2^{n+1}(n+1)\alpha^{n+1}. \tag{25}$$

Hence, proof is completed.

(IV) For $H_{m+1}^\alpha(x)$ is a solution of (18), we can write

$${}^{2}T_\alpha\left(H_{m+1}^\alpha(x)\right) - 2\alpha x^\alpha T_\alpha\left(H_{m+1}^\alpha(x)\right) + 2\alpha^2 m H_{m+1}^\alpha(x) = 0. \tag{26}$$

Using (II), we have

$$H_{m+1}^\alpha(x) = 2x^\alpha H_m^\alpha(x) - 2mH_{m-1}^\alpha(x). \tag{27}$$

(V) Using (21) and (27), the result is found.
(VI) We prove by induction. For $m=1$, the property is provided.

Assume the property is true for $m=n$. That is

$$H_n^\alpha(x) = (-\alpha^{-1})^n e^{x^{2\alpha}}\, {}^{n}T_\alpha\left(e^{-x^{2\alpha}}\right). \tag{28}$$

If (28) is substituted in the property (V), then

$$H_{n+1}^\alpha(x) = (-\alpha^{-1})^{n+1} e^{x^{2\alpha}}\, {}^{n+1}T_\alpha\left(e^{-x^{2\alpha}}\right)$$

is obtained. Hence, proof is completed.

(VII) The definition 2.1 is given for $x>0$. To avoid the problem of being undefined on $(-\infty,0]$, we assume $\alpha=\frac{1}{2j+1}$, with j any natural number. Since $H_m^\alpha(x)$ and $H_n^\alpha(x)$ are solution for the equation (18), respectively, then

$$ {}^{2}T_{\alpha}\left(H_{m}^{\alpha}(x)\right)-2\alpha x^{\alpha}T_{\alpha}\left(H_{m}^{\alpha}(x)\right)+2\alpha^{2}mH_{m}^{\alpha}(x)=0 \quad (29)$$

$$ {}^{2}T_{\alpha}\left(H_{m}^{\alpha}(x)\right)-2\alpha x^{\alpha}T_{\alpha}\left(H_{m}^{\alpha}(x)\right)+2\alpha^{2}mH_{m}^{\alpha}(x)=0 \quad (30)$$

Multiply (29) by $e^{-x^{2\alpha}}H_{n}^{\alpha}(x)$ and (30) by $e^{-x^{2\alpha}}H_{m}^{\alpha}(x)$ and subtract the resulting equation to get

$$T_{\alpha}\left[e^{-x^{2\alpha}}T_{\alpha}\left(H_{m}^{\alpha}(x)\right)\right]H_{n}^{\alpha}(x)-T_{\alpha}\left[e^{-x^{2\alpha}}T_{\alpha}\left(H_{n}^{\alpha}(x)\right)\right]H_{m}^{\alpha}(x)+2\alpha^{2}(m-n)e^{-x^{2\alpha}}H_{n}^{\alpha}(x)H_{m}^{\alpha}(x)=0 \quad (31)$$

If we apply the fractional integral to equation (31), then we get

$$\int_{-\infty}^{\infty}\left(T_{\alpha}\left[e^{-x^{2\alpha}}T_{\alpha}\left(H_{m}^{\alpha}(x)\right)\right]H_{n}^{\alpha}(x)-T_{\alpha}\left[e^{-x^{2\alpha}}T_{\alpha}\left(H_{n}^{\alpha}(x)\right)\right]H_{m}^{\alpha}(x)\right)d_{\alpha}x$$
$$+2\alpha^{2}(m-n)\int_{-\infty}^{\infty}e^{-x^{2\alpha}}H_{n}^{\alpha}(x)H_{m}^{\alpha}(x)d_{\alpha}x=0$$

If we apply fractional integration by parts to the first fractional integral in the above equation, we find the result of this fractional integral as zero. Hence proof is completed.

(VIII) The definition 2.1 is given for $x>0$. To avoid the problem of being undefined on $(-\infty,0]$, we assume $\alpha=\frac{1}{2j+1}$, with j any natural number. Let us define

$$I_{m,n}=\int_{-\infty}^{\infty}H_{m}^{\alpha}(x)\,H_{n}^{\alpha}(x)e^{-x^{2\alpha}}d_{\alpha}(x)$$

Then

$$I_{n-1,n+1}=\int_{-\infty}^{\infty}H_{n-1}^{\alpha}(x)\,H_{n+1}^{\alpha}(x)e^{-x^{2\alpha}}d_{\alpha}(x)=0$$

Using the property (IV), we get

$$\int_{-\infty}^{\infty}H_{n-1}^{\alpha}(x)\left(2x^{\alpha}H_{n}^{\alpha}(x)-2nH_{n-1}^{\alpha}(x)\right)e^{-x^{2\alpha}}d_{\alpha}(x)=0.$$

i.e.

$$\int_{-\infty}^{\infty}2x^{\alpha}H_{n-1}^{\alpha}(x)H_{n}^{\alpha}(x)\,e^{-x^{2\alpha}}d_{\alpha}(x)=2nI_{n-1,n-1}. \quad (32)$$

Recall that

$$H_{n}^{\alpha}(x)=(-\alpha^{-1})^{n}e^{x^{2\alpha}}\,{}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right).$$

Thus, equation (32) becomes

$$-(\alpha^{-1})^{2n-1}\int_{-\infty}^{\infty}2x^{\alpha}e^{x^{2\alpha}}\left({}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right)\right)\left({}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right)\right)d_{\alpha}(x)=2nI_{n-1,n-1}. \quad (33)$$

Note that

$$2x^{\alpha}e^{x^{2\alpha}}{}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right)=\frac{1}{\alpha}T_{\alpha}\left[e^{x^{2\alpha}}{}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right)\right]-\frac{1}{\alpha}e^{x^{2\alpha}}\,{}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right).$$

Then, equation (33) becomes

$$\int_{-\infty}^{\infty}(\alpha^{-1})^{2n}e^{x^{2\alpha}}\,{}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right)\,{}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right)d_{\alpha}(x)-(\alpha^{-1})^{2n}\int_{-\infty}^{\infty}T_{\alpha}\left[e^{x^{2\alpha}}{}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right)\right]{}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right)d_{\alpha}(x)$$
$$= 2nI_{n-1,n-1}$$

Using the property (IV) and integration by parts for conformable fractional derivative, we have

$$I_{n,n}-(\alpha^{-1})^{2n}\left[e^{x^{2\alpha}}{}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right){}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right)\right]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}(\alpha^{-1})^{2n}e^{x^{2\alpha}}{}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right){}^{n+1}T_{\alpha}\left(e^{-x^{2\alpha}}\right)d_{\alpha}(x)= 2nI_{n-1,n-1}.$$

We also have following equations:

$$\int_{-\infty}^{\infty}(\alpha^{-1})^{2n}e^{x^{2\alpha}}{}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right){}^{n+1}T_{\alpha}\left(e^{-x^{2\alpha}}\right)d_{\alpha}(x)=I_{n-1,n+1}=0,$$

$$-(\alpha^{-1})^{2n}\left[e^{x^{2\alpha}}{}^{n-1}T_{\alpha}\left(e^{-x^{2\alpha}}\right){}^{n}T_{\alpha}\left(e^{-x^{2\alpha}}\right)\right]_{-\infty}^{\infty}=0.$$

Hence, recurrence equation

$$I_{n,n}=2nI_{n-1,n-1}$$

is obtained. Repeating this operation *n* times yields the result

$$I_{n,n}=2^{n}n!\,I_{0,0}$$

where

$$I_{0,0}=\int_{-\infty}^{\infty}e^{-x^{2\alpha}}\,d_{\alpha}(x)=\frac{\sqrt{\pi}}{\alpha}$$

Hence proof is completed.

5. CONCLUSION

In this work, we give power series solutions around an ordinary point in homogenous case of sequential linear differential equation of conformable fractional of order 2α with variable coefficients. In addition, solving Hermite fractional differential equation, we obtain Hermite fractional polynomials for certain initial conditions. It is appeared that the results obtained in this work correspond to the results which are obtained in ordinary case.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Kilbas A, Srivastava H, Trujillo J. Theory and applications of fractional differential equations, North-Holland, New York; 2006.
2. Miller KS. An Introduction to Fractional Calculus and Fractional Differential Equations, J. Wiley and Sons, New York; 1993.
3. Podlubny I. Fractional differential equations, Academic Press, USA; 1999.
4. Xia ZN. Weighted Stepanov-like pseudoperiodicity and applications. Abstract and Applied Analysis. 2014;1-14.
5. Xia ZN. Asymptotically periodic solutions of semilinear fractional integro-differential equations. Advances in Difference Equations. 2014;1-9.
6. Lin Z, Wang JR, Wei W. Multipoint BVPs for generalized impulsive fractional differential equations. Applied Mathematics and Computation. 2015;258:608-616.
7. Lin Z, Wang JR, Wei W. Fractional differential equation models with pulses and criterion for pest management. Applied Mathematics and Computation. 2015;257: 398-408.
8. West B, Bologna M, Grigolini P. Physics of fractal operators. Springer, New York; 2003.
9. Hilfer R, eds. Applications of fractional calculus in physics. World Scientific, Singapore; 2000.
10. Tarasov VE. Fractional dynamics: Applications of fractional calculus to dynamics of particles, fields and media. Springer, New York; 2010.
11. Wu, Guo-Cheng, Dumitru Baleanu. Variational iteration method for the Burgers' flow with fractional derivatives—new Lagrange multipliers. Applied Mathematical Modelling. 2013;37(9):6183-6190.
12. Jafari Hossein, Hale Tajadodi, Dumitru Baleanu. A modified variational iteration

method for solving fractional Riccati differential equation by Adomian polynomials. Fractional Calculus and Applied Analysis. 2013;16(1):109-122.

13. Jafari H, Das S, Tajadodi H. Solving a multi-order fractional differential equation using homotopy analysis method. Journal of King Saud University-Science. 2011; 23(2):151-155.
14. Aghili A, Zeinali H. Solution to fractional schrödinger and airy differential equations via integral transforms. British Journal of Mathematics & Computer Science. 2014; 4(18):2630-2664.
15. Bao Siyuan, Deng Zi-chen. Fractional variational iteration method for fractional Fornberg-Whitham equation and comparison with the undetermined coefficient method. British Journal of Mathematics & Computer Science. 2015;6(3):187-203.
16. Merdan M, Gökdoğan A, Yıldırım A, Mohyud-Din ST. Numerical simulation of fractional Fornberg-Whitham equation by differential transformation method. Abstract and Applied Analysis. Hindawi Publishing Corporation; 2012.
17. Gökdoğan A, Mehmet M, Yildirim A. Adaptive multi-step differential transformation method to solving nonlinear differential equations. Mathematical and Computer Modelling. 2012;55(3):761-769.
18. Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. J. Comput. Appl. Math. 2014;264:65-70.
19. Abdeljawad T. On conformable fractional calculus. J. Comput. Appl. Math. 2014;279: 57-66.
20. Khalil R, Mamoon AH. Legendre fractional differential equation and Legendre fractional polynomials. International Journal of Applied Mathematical Research. 2014;3(3):214-219.
21. Batarfi H, et al. Three-point boundary value problems for conformable fractional differential equations. Journal of Function Spaces; 2015.
22. Anderson DR, Richard IA. Fractional-order boundary value problem with Sturm-Liouville boundary conditions. arXiv preprint arXiv:1411.5622; 2014.
23. Bell WW. Special functions for scientists and engineers, London, VanNostrand; 1968.

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