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Orthogonal Double Covers of Complete Bipartite Graphs by A Special Class of Disjoint Union of Path and A Complete Bipartite Graph

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Research Article

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Abstract

Let H be a graph on n vertices and let $\mathcal{G}$ be a collection of n subgraphs of H, one for each vertex, $\mathcal{G}$ is an orthogonal double cover (ODC) of H if every edge of H is contained in exactly two members of $\mathcal{G}$ and any two members share an edge whenever the corresponding vertices are adjacent in H and share no edges whenever the corresponding vertices are non-adjacent in H. In this paper, we are concerned with the symmetric starter vectors of the orthogonal double covers of the complete bipartite graphs and using this method to construct ODCs by the disjoint union of path and a complete bipartite graph. Here, we consider P_m the path on m vertices where $4 \leq m \leq 11$.

Keywords: Graph decomposition; Orthogonal double cover; Symmetric starter.

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1 Introduction

For the definition of an orthogonal double cover (ODC) of the complete graph K_n by a graph G and for a survey on this topic, see (1). In (2) this concept has been generalized to ODCs of any graph H by a graph G.

While in principle any regular graph is worth considering (e.g., the remarkable case of hypercubes has been investigated in (2)), the choice of $H = K_{n,n}$ is quite natural, also in view of a technical motivation: ODCs of such graphs are a helpful tool for constructing ODCs of K_n(see(3), p. 48).

In this paper, we assume that $H = K_{n,n}$, the complete bipartite graph with partition sets of size n each.

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An ODC of $K_{n,n}$ is a collection $G = \{G_0, G_1, \ldots, G_{n-1}, F_0, F_1, \ldots, F_{n-1}\}$ of $2n$ subgraphs (called pages) of $K_{n,n}$ such that

(i) every edge of $K_{n,n}$ is in exactly one page of $\{G_0, G_1, \ldots, G_{n-1}\}$ and in exactly one page of $\{F_0, F_1, \ldots, F_{n-1}\}$;

(ii) for $i,\ j \in \{0, 1, 2, \ldots, n-1\}$ and $i \neq j,\ E(G_i) \cap E(G_j) = E(F_i) \cap E(F_j) = \emptyset$; and $|E(G_i) \cap E(F_j)| = 1$ for all $i,\ j \in \{0, 1, 2, \ldots, n-1\}$.

If all the pages are isomorphic to a given graph G, then G is said to be an ODC of $K_{n,n}$ by G.

Denote the vertices of the partition sets of $K_{n,n}$ by $\{0_0, 1_0, \ldots, (n-1)_0\}$ and $\{0_1, 1_1, \ldots, (n-1)_1\}$. The length of an edge x_0y_1 of $K_{n,n}$ is defined to be the difference $y - x$, where $x,\ y \in \mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Note that sums and differences are calculated in $\mathbb{Z}_n$ (that is, sums and differences are calculated modulo n).

Throughout the paper we make use of the usual notation: $K_{m,n}$ for the complete bipartite graph with partition sets of sizes m and n, P_n for the path on n vertices, K_1 for an isolated vertex, $G \cup H$ for the disjoint union of G and H, and mG for m disjoint copies of G.

An algebraic construction of ODCs via “symmetric starters” (see Section 2) has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leq 9$: a few exceptions apart, all graphs G are found by this way (see (3), Table 1). This method has been applied in (3) to detect some infinite classes of graphs G for which there are ODCs of $K_{n,n}$ by G. El shanawany and et al (4) studied the orthogonal double covers of $K_{n,n}$ by $P_{m+1} \cup^* S_{n-m}$, where n and m are integers, $2 \leq m \leq 10$, $m \leq n$ and $P_{m+1} \cup^* S_{n-m}$ is a tree obtained from the path P_{m+1} with m edges and a star S_{n-m} with $n-m$ edges by identifying an end-vertex of P_{m+1} with the center of S_{n-m}. Much of research on this subject focused on the detection of ODCs with pages isomorphic to a given graph G. For a summary of results on ODCs, see (1; 5). The other terminologies not defined here can be found in (6) .

The paper is organized as follows. Section 2 describes the technique that will be used throughout this paper. Section 3 offers some insights into the case on ODC of the complete bipartite graphs by a special class of disjoint union of path and a complete bipartite graph.

1.1 Symmetric Starters

All graphs here are finite, simple and undirected. Let $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\}$ be an (additive) abelian group of order n. The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v, i) \in \Gamma \times \mathbb{Z}_2$ we will write v_i for the corresponding vertex and define $\{w_i, u_j\} \in E(K_{n,n})$ if and only if $i \neq j$, for all $w,\ u \in \Gamma$ and $i,\ j \in \mathbb{Z}_2$. If there is no chance of confusion (w, u) will be written instead of $\{w_0, u_1\}$ for the edge between the vertices $w_0,\ u_1$.

Let G be a spanning subgraph of $K_{n,n}$ and let $a \in \Gamma$. Then the graph $G + a$ with $E(G + a) = \{(u + a, v + a) : (u, v) \in E(G)\}$ is called the a-translate of G. The length of an edge $e = (u, v) \in E(G)$ is defined by $d(e) = v - u$.

G is called a half starter with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are mutually distinct, i.e. $\{d(e) : e \in E(G)\} = \Gamma$. The following three results were established in (3).

Theorem 1.1. *If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$ i.e. $\cup_{a\in\Gamma}E(G+a) = E(K_{n,n})$.*

Hereafter, a half starter G will be represented by the vector $v(G) = (v_{\gamma_0}, \ldots, v_{\gamma_{n-1}})$, where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i in G.

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 1.2. *If two half starter vectors $v(G_0)$ and $v(G_1)$ are orthogonal, then $G = \{G_{a,i} : (a, i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = G_i + a$ is an ODC of $K_{n,n}$.*

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{(u_0, v_1) : (v_0, u_1) \in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a symmetric starter with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 1.3. *Let n be a positive integer and let G be a half starter represented by the vector $v(G) = (v_{\gamma_0}, \ldots, v_{\gamma_{n-1}})$. Then G is a symmetric starter if and only if $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.*

2 Main Results

In the following, if there is no danger of ambiguity, w_iu_j will be written instead of $\{w_i, u_j\}$ for the edge between the vertices w_i, u_j where $i,\ j \in \mathbb{Z}_2$.

Theorem 2.1. *Let $m, p, n > 3$ be positive integers such that $mp = n - 3$. Then there is a symmetric starter vector of an ODC of $K_{n,n}$ by $G = P_4 \cup K_{m,p} \cup (2n - (4 + m + p)K_1$.*

Proof. For a positive integer $n > 3$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i = n - 1$, $v_i = x_0$ if $2 \le i \le p+1$, $v_i = x_1$ if $p+2 \le i \le 2p+1, \ldots,$ $v_i = x_{m-1}$ if $(m-1)p+2 \le i \le mp+1$. Where $x_j = 1 - jp$; $0 \le j \le m-1$. By definition of $v(G)$, for any $i \in \mathbb{Z}_n,$ the i^{th}graph is isomorphic to the graph $G = P_4 \cup K_{m,p}$ has edges $E(G) = \{(0+i)_1(0+i)_0,\ (0+i)_0(1+i)_1,\ (1+i)_1(2+i)_0\}$ $\cup_{j=3}^{p+2}\{(x_\alpha + i)_0(j+i)_1 : 0 \le \alpha \le m-1\}$ and hence $G \cong P_4 \cup K_{m,p} \cup (2n - (4 + m + p)K_1$. For $i = 0$, $v_i - v_{-i} + i = 0$, for $i \in \{1, n-1\}$, $v_i - v_{-i} + i = -i$ and for $jp + 2 \le i \le (j+1)p + 1$, $v_i - v_{-i} + i = x_j - x_{m-(j+1)} + i$. By theorem 1.3, $v(G)$ is a symmetric starter vector. □

Lemma 2.2. *Let $n > 3$ be a positive integer. Then there is a symmetric starter vector of an ODC of $K_{n,n}$ by $G = P_4 \cup K_{1,n-3} \cup (n-2)K_1$.*

Proof. For a positive integer $n > 3$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i = n - 1$ and $v_i = 1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_n,$ the i^{th}graph is isomorphic to the graph $G = P_4 \cup K_{1,n-3}$ has edges $E(G) = \{(0+i)_1(0+i)_0,\ (0+i)_0(1+i)_1,\ (1+i)_1(2+i)_0\} \cup_{j=3}^{n-1} \{1 + i)_0(j+i)_1\}$ and hence $G \cong P_4 \cup K_{1,n-3} \cup (n-2)K_1$. For $i \in \{1, n-1\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector. □

Theorem 2.3. *Let t, n be positive integers such that $1 \le t \le 10$ and $t < n$. Then there is a symmetric starter vector of an ODC of $K_{n,n}$ by $G = P_{t+1} \cup K_{1,n-t} \cup (n-2)K_1$.*

Proof. For $t \in \{1, 2\}$, the theorem was already proved using direct construction in (3). For $t = 3$, see lemma 2.2. In what follows we find a suitable symmetric strater vector of $\mathbb{Z}_n$ in each of the remaining cases:

Case 1. $t = 4$

For $n = 2m$ and $m > 2$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, m\}$, $v_i = 2m - 1$ if $i \in \{1, 2m-1\}$ and $v_i = 2m-1-i$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the i^{th}graph is isomorphic to the graph $G = P_5 \cup K_{1,2m-4}$ has edges $E(G) = \{(2m-2+i)_1(2m-1+i)_0, (2m-1+i)_0(0+i)_1, (0+i)_1(0+i)_0, (0+i)_0(m+i)_1\} \cup \{(2m-1+i)_1(j+i)_0 : 1 \leq j \leq m-2, m \leq j \leq 2m-3\}$ and hence $G \cong P_5 \cup K_{1,2m-4} \cup (2m-2)K_1$. For $i \in \{0, 1, m, 2m-1\}$, $v_i - v_{-i} + i = i$ and for otherwise, $v_i - v_{-i} + i = -i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

For $n = 2m+1$ and $m > 2$, define the vector $v(G)$ as $v_i = 0$ if $i = 1$, $v_i = 2$ if $i \in \{m, 2m\}$, $v_i = 1$ if $i = m+1$ and $v_i = m+1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m+1,}$ the i^{th}graph is isomorphic to the graph $G = P_5 \cup K_{1,2m-3}$ has edges $E(G) = \{(0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0\} \cup \{(m+1+i)_0(j+i)_0 : 2 \leq j \leq m-1, \ j = m+1, m+3 \leq j \leq 2m\}$ and hence $G \cong P_5 \cup K_{1,2m-3} \cup (2m-1)K_1$. For $i \in \{1, m, m+1, 2m\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

Case 2. $t = 5$

For $n = 2m$ and $m > 2$, define the vector $v(G)$ as $v_i = m-1$ if $i \in \{1, 2m-1\}$, $v_i = 0$ if $i \in \{m-1, m\}$, $v_i = 2m-2$ if $i = m+1$ and $v_i = 2m-1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the i^{th}graph is isomorphic to the graph $G = P_6 \cup K_{1,2m-5}$ has edges $E(G) = \{(2m-2+i)_0(m-1+i)_1, (m-1+i)_1(0+i)_0, (0+i)_0(m+i)_1, (m+i)_1(m-1+i)_0, (m-1+i)_0(m-2+i)_1\} \cup \{(2m-1+i)_0(j+i)_1 : 1 \leq j \leq m-3, \ j = 2m-1, \ m+1 \leq j \leq 2m-3\}$ and hence $G \cong P_6 \cup K_{1,2m-5} \cup (2m-2)K_1$. For $i \in \{m-1, m+1\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

For $n = 2m+1$ and $m > 2$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i \in \{m, 2m\}$, $v_i = 1$ if $i = m+1$ and $v_i = m+1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m+1,}$ the i^{th}graph is isomorphic to the graph $G = P_6 \cup K_{1,2m-4}$ has edges $E(G) = \{(0+i)_1(0+i)_0, (0+i)_0(1+i)_1, (1+i)_1(2+i)_{0,} (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0\} \cup \{(m+1+i)_0(j+i)_1 : 2 \leq j \leq m-1, \quad m+3 \leq j \leq 2m\}$ and hence $G \cong P_6 \cup K_{1,2m-4} \cup (2m-1)K_1$. For $i \in \{1, m, m+1, 2m\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

Case 3. $t = 6$

For $n = 2m$ and $m > 3$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i \in \{m, 2m-1\}$, $v_i = 1$ if $i \in \{m-1, m+1\}$ and $v_i = m+1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the i^{th}graph is isomorphic to the graph $G = P_7 \cup K_{1,2m-6}$ has edges $E(G) = \{(0+i)_1(0+i)_0, (0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1\} \cup \{(m+1+i)_0(j+i)_1 : 3 \leq j \leq m-1, \quad m+3 \leq j \leq 2m-1\}$ and hence $G \cong P_7 \cup K_{1,2m-6} \cup (2m-2)K_1$. For $i \in \{1, 2m-1\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

For $n = 2m+1$ and $m > 2$, define the vector $v(G)$ as $v_i = 0$ if $i = 1, v_i = 1$ if $i \in \{m-1, m+1\}$, $v_i = 2$ if $i \in \{m, 2m\}$, $v_i = 2m-1$ if $i = m+2$ and $v_i = m+1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m+1,}$ the i^{th}graph is isomorphic to the graph $G = P_7 \cup K_{1,2m-5}$ has edges $E(G) = \{(0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1, (m+i)_1(2m-1)_0\} \cup \{(n+1+i)_0(j+i)_1 : 3 \leq j \leq m-1, \ j = m+1, \ m+3 \leq j \leq 2m-1\}$ and hence $G \cong P_7 \cup K_{1,2m-5} \cup (2m-1)K_1$. For $i \in \{1, m-1, m, m+1, m+2, 2m\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

Case 4. $t = 7$

For $n = 2m$ and $m > 3$, define the vector $v(G)$ as $v_i = 0$ if $i = 1$, $v_i = 2$ if $i \in \{m, 2m-1\}$, $v_i = 1$ if $i \in \{m-1, m+1\}$, $v_i = m+2$ if $i \in \{2, 2m-2\}$ and $v_i = m+1-i$ otherwise.

By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the i^{th}graph is isomorphic to the graph $G = P_8 \cup K_{1,2m-7}$ has edges $E(G) = \{ (0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1, (m+i)_1(m+2)_0, (m+2)_0\ (m+4)_1\} \cup \{(m+1+i)_1(j+i)_0 : 3 \le j \le m-2,\ j = m+1, m+4 \le j \le 2m-1\}$ and hence $G \cong P_8 \cup K_{1,2m-7} \cup (2m-2)K_1$. For $i \in \{2, m-1, m, m+1, 2m-2\}$, $v_i - v_{-i} + i = i$ and for otherwise, $v_i - v_{-i} + i = -i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

For $n = 2m + 1$ and $m > 3$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i \in \{m, 2m\}$, $v_i = 1$ if $i \in \{m-1, m+1\}$, $v_i = 2m-1$ if $i = m+2$ and $v_i = m+1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m+1,}$ the i^{th}graph is isomorphic to the graph $G = P_8 \cup K_{1,2m-6.}$ has edges $E(G) = \{(0+i)_1(0+i)_0, (0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1, (m+i)_1(2m-1+i)_0\} \cup \{(m+1+i)_0(j+i)_1 : 3 \le j \le m-1,\ m+3 \le j \le 2m-1\}$ and hence $G \cong P_8 \cup K_{1,2m-6.} \cup (2m-1)K_1$. For $i \in \{1, m-1, m, m+1, m+2, 2m\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

Case 5. $t = 8$

For $m > 4$ and $n = 2m$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i \in \{m, 2m-1\}$, $v_i = 1$ if $i \in \{m-1, m+1\}$, $v_i = m+2$ if $i \in \{2, 2m-2\}$ and $v_i = m+1-i$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the i^{th}graph is isomorphic to the graph $G = P_9 \cup K_{1,2m-8}$ has edges $E(G) = \{(0+i)_1(0+i)_0, (0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1, (m+i)_1(m+2+i)_0, (m+2+i)_0((m+4+i)_1\} \cup \{(m+1+i)_1(j+i)_0\} : 3 \le j \le m-2,\ \ m+4 \le j \le 2m-1\}$ and hence $G \cong P_9 \cup K_{1,2m-8} \cup (2m-2)K_1$. For $i \in \{0, 2, m-1, m, m+1, 2m-2\}$, $v_i - v_{-i} + i = i$ and for otherwise, $v_i - v_{-i} + i = -i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

Case 6. $t = 9$

For $n = 2m$ and $m > 6$, define the vector $v(G)$ as $v_i = 0$ if $i = 1$, $v_i = 2$ if $i \in \{m, 2m-1\}$, $v_i = 1$ if $i \in \{m-1, m+1\}$, $v_i = m-2$ if $i \in \{2, 2m-2\}$, $v_i = 2m-7$ if $i \in \{m-3, m+3\}$ and $v_i = 2m-1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the i^{th}graph is isomorphic to the graph $G = P_{10} \cup K_{1,2m-9}$ has edges $E(G) = \{ (0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1, (m+i)_1(m-2)_0, (m-2)_0\ (m-4)_1, (m-4)_1(2m-7+i)_0, (2m-7+i)_0(m-10)_1\} \cup \{(2m-1+i)_0(j+i)_1 : 2 \le j \le m-5,\ m+3 \le j \le 2m-4, j \in \{m-3, m+1, 2m-1\}\}$ and hence $G \cong P_{10} \cup K_{1,2m-9} \cup (2m-2)K_1$. For $i \in \{1, 2m-1\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

For $n = 2m+1$ and $m > 4$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i \in \{m, 2m\}$, $v_i = 1$ if $i \in \{m-1, m+1\}$, $v_i = 2m-1$ if $i \in \{m+2, m+3\}$, $v_i = 3$ if $i = m-2$ and $v_i = m+1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m+1,}$ the i^{th}graph is isomorphic to the graph $G = P_{10} \cup K_{1,2m-8}$ has edges $E(G) = \{(0+i)_1(0+i)_0, (0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1, (m+i)_1(2m-1+i)_0, (2m-1+i)_0(m+1+i)_1, (m+1+i)_1(3+i)_0\} \cup \{(m+1+i)_0(j+i)_1 : 4 \le j \le m-1,\ \ m+3 \le j \le 2m-2\}$ and hence $G \cong P_{10} \cup K_{1,2m-8} \cup (2m-1)K_1$. For $i \in \{1,\ m-2,\ m-1,\ m,\ m+1,\ m+2,\ m+3,\ 2m\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector.

Case 7. $t = 10$

For $n = 2m$ and $m > 5$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0, 1\}$, $v_i = 2$ if $i \in \{m, 2m-1\}$, $v_i = 1$ if $i \in \{\ m-1, m+1\}$, $v_i = m+2$ if $i \in \{2,\ 2m-2\}$, $v_i = m+1$ if $i \in \{3,\ 2m-3\}$ and $v_i = m+1-i$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2m,}$ the i^{th}graph is isomorphic to the graph $G = P_{11} \cup K_{1,2m-10}$ has edges $E(G) = \{(0+i)_1(0+i)_0,\ (0+i)_0(1+i)_1, (1+i)_1(2+i)_0, (2+i)_0(m+2+i)_1, (m+2+i)_1(1+i)_0, (1+i)_0(m+i)_1, (m+i)_1(m+2+i)_0, (m+2+i)_0(m+4+i)_1, (m+4+i)_1(m+1+i)_0, (m+1+i)_0(m-2+i)_1\} \cup \{(m+1+i)_0(j+i)_1 : 3 \le j \le m-3,\ \ m+5 \le j \le 2m-1\}$ and hence $G \cong P_{11} \cup K_{1,2m-10} \cup (2m-2)K_1$. For $i \in \{0, 2, 3, m-1, m, m+1, 2m-3, 2m-2\}$, $v_i - v_{-i} + i = i$ and for otherwise, $v_i - v_{-i} + i = -i$. By theorem 1.3, $v(G)$ is a symmetric starter vector. □

Theorem 2.4. *Let $n > 3$ be a positive integer. Then there is a symmetric starter vector of an ODC of $K_{2n,2n}$ by $G = 2P_4 \cup K_{1,2n-6} \cup (2n-3)K_1$.*

Proof. For a positive integer $n > 3$, define the vector $v(G)$ as $v_i = 0$ if $i \in \{0,1\}$, $v_i = 2$ if $i = 2n-1$, $v_i = 1$ if $i \in \{n, n+1\}$, $v_i = 3$ if $i = n-1$ and $v_i = n+1$ otherwise. By definition of $v(G)$, for any $i \in \mathbb{Z}_{2n}$, the i^{th}graph is isomorphic to the graph $G = 2P_4 \cup K_{1,2n-6}$ has edges $E(G) = \{(0+i)_1(0+i)_0, (0+i)_0(1+i)_1, (1+i)_1(2+i)_0\} \cup \{(3+i)_0(n+2+i)_1, (n+2+i)_1(1+i)_0, (1+i)_0(n+1+i)_1\} \cup \{(n+1+i)_0(j+i)_1 : 3 \leq j \leq n-1, \quad n+3 \leq j \leq 2n-1\}$ and hence $G \cong 2P_4 \cup K_{1,2n-6} \cup (2n-3)K_1$. For $i \in \{1, n-1, n+1, 2n-1\}$, $v_i - v_{-i} + i = -i$ and for otherwise, $v_i - v_{-i} + i = i$. By theorem 1.3, $v(G)$ is a symmetric starter vector. □

3 Conclusions

In conclusion, we conjecture that if m, n are positive integers and $n > m$, there is a symmetric starter vector of an ODC of a complete bipartite graph $K_{n,n}$ by the disjoint union of P_{m+1} and $K_{1,n-m}$.

We can summrize our results in the following table

H	G
$K_{n,n}$	$P_4 \cup K_{m,p} \cup (2n-(4+m+p)K_1$
$K_{n,n}$	$P_5 \cup K_{1,n-4} \cup (n-2)K_1$
$K_{n,n}$	$P_6 \cup K_{1,n-5} \cup (n-2)K_1$
$K_{n,n}$	$P_7 \cup K_{1,n-6} \cup (n-2)K_1$
$K_{n,n}$	$P_8 \cup K_{1,n-7}. \cup (n-2)K_1$
$K_{2n,2n}$	$P_9 \cup K_{1,2n-8} \cup (2n-2)K_1$
$K_{n,n}$	$P_{10} \cup K_{1,n-9} \cup (n-2)K_1$
$K_{2n,2n}$	$P_{11} \cup K_{1,2n-10} \cup (2n-2)K_1$
$K_{2n,2n}$	$2P_4 \cup K_{1,2n-6} \cup (2n-3)K_1$

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Competing Interests

The authors declare that no competing interests exist.

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