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(R,S)-(Skew) Symmetric Solutions to Matrix Equation $AXB = C$ over Quaternions

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Abstract: (R,S)-(skew) symmetric matrices have numerous applications in civil engineering, information theory, numerical analysis, etc. In this paper, we deal with the (R,S)-(skew) symmetric solutions to the quaternion matrix equation $AXB = C$. We use a real representation A^{τ} to obtain the necessary and sufficient conditions for $AXB = C$ to have (R,S)-(skew) symmetric solutions and derive the solutions when it is consistent. We also derive the least-squares (R,S)-(skew) symmetric solution to the above matrix equation.

Keywords: quaternion matrix equation; (R,S)-(skew) symmetric; real representation

MSC: 15A24; 15b33; 15B57

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1. Introduction

A real quaternion (also called Hamilton quaternion) is usually expressed as

$$\mathbb{H} = \left\{ a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

It is a four-dimensional division algebra over the real number field $\mathbb{R}$. During the past decades, matrices over quaternions have played important roles in signal processing, aerospace, color image processing, and many other areas [1–6].

Recall that an involutory matrix R is a square matrix satisfying $R^2 = I$, an identity matrix. Clearly, $\pm I$ are trivial involutory matrices. A matrix $A \in \mathbb{C}^{m\times n}$ is said to be an (R,S)-symmetric (resp. (R,S)-skew symmetric) matrix if there exist nontrivial involution matrices R and S such that $RAS = A$ (resp. $RAS = -A$) (see [7,8]).

(R,S)-(skew) symmetric matrices are widely used in linear system theory, numerical analysis, and physics [9–14]. Trench [8] discussed the (R,S)-(skew) symmetric solutions to the complex matrix equation $XB = C$. Dehghan and Hajarian [7] also considered the (R,S)-(skew) symmetric solutions to some complex matrix equations. They derived the solvability conditions for the existence of solutions and obtained the general solutions when the matrix equations are solvable. Zhang and Wang [15] derived the (R,S)-(skew) symmetric extreme rank solutions to the quaternion system $AX = B, XC = D$. Some special cases of (R,S)-(skew) symmetric matrices, such as (anti-)centrosymmetric matrices, P-(skew) symmetric matrices, generalized reflexive matrix, and reflexive (antireflexive) matrices, have been discussed [13,14,16–18]. For instance, Zhou et al. [18] studied the problems of centrosymmetric matrices over $\mathbb{C}$, where R, S are the J, which are on the secondary diagonal and zeros elsewhere. Wang et al. [17] considered the P-(skew) symmetric solutions to a pair of quaternion matrix equations.

The matrix equation $AXB = C$ is of substantial research significance due to its wide range of applications and its relevance in solving fundamental problems in various disciplines. The (R, S)-(skew) symmetric matrices include many kinds of important special matrices, like centrosymmetric matrices, P-(skew) symmetric, generalized reflexive matrix, and reflexive (antireflexive) matrices; these kinds of solutions have far-reaching implications in areas ranging from mathematics and engineering to computer science and data analysis [13,14,16–18].

Motivated by the work and research significance mentioned above, we consider the (R, S)-(skew) symmetric solutions and least-squares (R, S)-(skew) symmetric solutions to the quaternion matrix equation

$$AXB = C, \tag{1}$$

where $A \in \mathbb{H}^{p\times m}, B \in \mathbb{H}^{n\times q}, C \in \mathbb{H}^{p\times q}$ are known matrices and $X \in \mathbb{H}^{m\times n}$ is an unknown matrix.

The paper is organized as follows: In Section 2, we first introduce some preliminary results. Then we derive the solvability conditions of the equation $AXB = C$ for (R, S)-(skew) symmetric solutions. In Section 3, the least-squares (R, S)-(skew) symmetric solutions are given. Finally, we provide a numerical example in Section 4.

Throughout this paper, we propose some notations. For a matrix A, A^T, A^*, and $r(A)$ denote the transpose, conjugate transpose, and rank of A separately. Moreover, $A^\dagger$ stands for the Moore–Penrose inverse of A. I_n will be the $n \times n$ identity matrix.

2. The Solvability

The real representation method is one of the standard and efficient ways to solve questions over quaternions. There are several real representations; see, for example, [19–21]. In this paper, we will use the following X^τ:

For $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{H}^{m\times n},\ X_i \in \mathbb{R}^{m\times n}, i = 1, 2, 3, 4,$

$$X^\tau = \begin{bmatrix} X_1 & -X_2 & -X_3 & -X_4 \\ X_2 & X_1 & -X_4 & X_3 \\ X_3 & X_4 & X_1 & -X_2 \\ X_4 & -X_3 & X_2 & X_1 \end{bmatrix}.$$

It is easy to verify that this real representation can convert an (R, S)-(skew) symmetric matrix into a real (R^τ, S^τ)-(skew) symmetric matrix. For further discussions of our problem, we introduce the following orthogonal matrices:

$$Q_n = \begin{bmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{bmatrix}, \quad G_n = \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ -I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \end{bmatrix},$$

$$T_n = \begin{bmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{bmatrix}.$$

Now, we summarize some properties of the above real representation in the following lemma:

Lemma 1. *Let $A, B \in \mathbb{H}^{m\times n}$, $C \in \mathbb{H}^{n\times s}$, $a \in \mathbb{R}$. Then*

(a) $(A + B)^\tau = A^\tau + B^\tau$, $(aA)^\tau = aA^\tau$;
(b) $(AC)^\tau = A^\tau C^\tau$;
(c) $Q_m^T A^\tau Q_n = A^\tau$, $G_m^T A^\tau G_n = A^\tau$, $T_m^T A^\tau T_n = A^\tau$;
(d) $(A^*)^\tau = (A^\tau)^T$, $\left(A^{-1}\right)^\tau = (A^\tau)^{-1}$;
(e) *Let $A \in \mathbb{H}^{m\times m}$. Then A^τ commutes with Q_m, G_m, T_m.*

Definition 1. *Assume that $R \in \mathbb{H}^{m\times m}$ and $S \in \mathbb{H}^{n\times n}$ are nontrivial involutory matrices. $A \in \mathbb{H}^{m\times n}$ is an (R,S)-symmetric (resp. (R,S)-skew symmetric) matrix when A satisfies $RAS = A$ (resp. $RAS = -A$).*

Now, we are in a position to derive some necessary and sufficient conditions for the matrix Equation (1) to have (R,S)-symmetric (resp. (R,S)-skew symmetric) solutions and provide the solutions when the matrix equation is consistent. Let $R \in \mathbb{H}^{m\times m}$ and $S \in \mathbb{H}^{n\times n}$ be nontrivial involutory matrices. By (d) of Lemma 1,

$$R^\tau = (R^{-1})^\tau = (R^\tau)^{-1} \neq \pm I,$$

$$S^\tau = (S^{-1})^\tau = (S^\tau)^{-1} \neq \pm I;$$

thus, R^τ, S^τ are also nontrivial involutory matrices. For R^τ and S^τ, according to [8], we can find positive numbers r, k and matrices $P \in \mathbb{C}^{4m\times r}, Q \in \mathbb{C}^{4m\times(4m-r)}, U \in \mathbb{C}^{4n\times k}, V \in \mathbb{C}^{4n\times(4n-k)}$ such that

$$P^*P = I, \quad Q^*Q = I, \quad R^\tau P = P, \quad R^\tau Q = -Q,$$

and

$$U^*U = I, \quad V^*V = I, \quad S^\tau U = U, \quad S^\tau V = -V.$$

Next, we denote

$$\widehat{U} = \frac{U^*(I+S^\tau)}{2}, \widehat{V} = \frac{V^*(I-S^\tau)}{2}, \widehat{P} = \frac{P^*(I+R^\tau)}{2}, \widehat{Q} = \frac{Q^*(I-R^\tau)}{2}. \tag{2}$$

For $A^\tau \in \mathbb{R}^{4p\times 4m}$ and $B^\tau \in \mathbb{R}^{4n\times 4q}$ of (1), we perform the following decomposition:

$$A^\tau \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}, \quad \mathcal{A}_1 \in \mathbb{C}^{4p\times r}, \mathcal{A}_2 \in \mathbb{C}^{4p\times(4m-r)},$$

$$\begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} B^\tau = \begin{bmatrix} \mathcal{B}_1{}^T & \mathcal{B}_2{}^T \end{bmatrix}^T, \quad \mathcal{B}_1 \in \mathbb{C}^{k\times 4q}, \mathcal{B}_2 \in \mathbb{C}^{(4n-k)\times 4q}.$$

Now, we have our main results as follows:

Theorem 1. *Let $A \in \mathbb{H}^{p\times m}$, $B \in \mathbb{H}^{n\times q}$, $C \in \mathbb{H}^{p\times q}$. Then, there are three equivalent statements:*

(a) *The matrix Equation (1) has an (R,S)-symmetric solution $X \in \mathbb{H}^{m\times n}$;*

(b) *The matrix equation*

$$A^\tau Y B^\tau = C^\tau \tag{3}$$

has an (R^τ, S^τ)-symmetric solution $Y \in \mathbb{R}^{4m\times 4n}$;

(c) *The following rank equalities hold:*

$$r\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 & C^\tau \end{bmatrix} = r\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}, \quad r\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ C^\tau \end{bmatrix} = r\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix},$$

$$r\begin{bmatrix} C^\tau & \mathcal{A}_1 \\ \mathcal{B}_2 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & \mathcal{A}_1 \\ \mathcal{B}_2 & 0 \end{bmatrix}, \quad r\begin{bmatrix} C^\tau & \mathcal{A}_2 \\ \mathcal{B}_1 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & \mathcal{A}_2 \\ \mathcal{B}_1 & 0 \end{bmatrix}.$$

Furthermore, if the matrix Equation (1) is solvable, then

$$X = \frac{1}{16}\begin{bmatrix} I_m \\ iI_m \\ jI_m \\ kI_m \end{bmatrix}^T \left(Y + Q_m Y Q_n{}^T + G_m Y G_n{}^T + T_m Y T_n{}^T\right) \begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix} \tag{4}$$

is an (R, S)-symmetric solution to (1), where

$$Y = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} Y_{PU} & 0 \\ 0 & Y_{QV} \end{bmatrix} \begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} \tag{5}$$

with

$$Y_{PU} = \widetilde{Y}_{PU} + S_1 F_J U E_H T_1 + F_{\mathcal{A}_1} V_1 + V_2 E_{\mathcal{B}_1},$$
$$Y_{QV} = \widetilde{Y}_{QV} + S_2 F_J U E_H T_2 + F_{\mathcal{A}_2} W_1 + W_2 E_{\mathcal{B}_2},$$
$$S_1 = \begin{bmatrix} I_r & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & I_{4m-r} \end{bmatrix}, T_1 = \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$
$$T_2 = \begin{bmatrix} 0 \\ I_{4n-k} \end{bmatrix}, J = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}, H = \begin{bmatrix} \mathcal{B}_1 \\ -\mathcal{B}_2 \end{bmatrix};$$

U, V_1, V_2, W_1, W_2 are arbitrary, and $\widetilde{Y}_{PU}, \widetilde{Y}_{QV}$ are particular solutions to the matrix equation $\mathcal{A}_1 Y_{PU} \mathcal{B}_1 + \mathcal{A}_2 Y_{QV} \mathcal{B}_2 = C^\tau$.

Proof. Let $R \in \mathbb{H}^{m\times m}$ and $S \in \mathbb{H}^{n\times n}$. For the matrix Equation (1), we convert it into the matrix equation $A^\tau Y B^\tau = C^\tau$ by the real representation method. First, it can be shown that each (R^τ, S^τ)-symmetric solution Y of the real matrix Equation (3) can generate an (R, S)-symmetric solution X of the original matrix Equation (1).

Suppose (3) has an (R^τ, S^τ)-symmetric solution Y, i.e., $R^\tau Y S^\tau = Y$. Y may not have the structure of a real representation; thus, we need to construct $\mathcal{Y}$ with the structure of a real representation from Y. According to Lemma 1, we have the following three equations:

$$\left(Q_p{}^T A^\tau Q_m\right) Y \left(Q_n{}^T B^\tau Q_q\right) = Q_p{}^T C^\tau Q_q,$$
$$\left(G_p{}^T A^\tau G_m\right) Y \left(G_n{}^T B^\tau G_q\right) = G_p{}^T C^\tau G_q,$$
$$\left(T_p{}^T A^\tau T_m\right) Y \left(T_n{}^T B^\tau T_q\right) = T_p{}^T C^\tau T_q.$$

Since $Q_p, Q_n, G_p, G_n, T_p, T_n$ are nonsingular,

$$A^\tau (Q_m Y Q_n^T) B^\tau = C^\tau,$$
$$A^\tau (G_m Y G_n^T) B^\tau = C^\tau,$$
$$A^\tau (T_m Y T_n^T) B^\tau = C^\tau,$$

and by Lemma 1(e), we have

$$R^\tau Q_m Y Q_n^T S^\tau = Q_m R^\tau Y S^\tau Q_n^T = Q_m Y Q_n^T,$$
$$R^\tau G_m Y G_n^T S^\tau = G_m R^\tau Y S^\tau G_n^T = G_m Y G_n^T,$$
$$R^\tau T_m Y T_n^T S^\tau = T_m R^\tau Y S^\tau T_n^T = T_m Y T_n^T.$$

Therefore, $Q_m Y Q_n^T$, $G_m Y G_n^T$, and $T_m Y T_n^T$ are also the (R^τ, S^τ)-symmetric solutions of (3), and so is

$$\mathcal{Y} = \frac{1}{4}\left(Y + Q_m Y Q_n^T + G_m Y G_n^T + T_m Y T_n^T\right). \tag{6}$$

Suppose that Y can be written in the form of a block matrix:

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix}$$

and substitute it in (6). Then by computation, we have

$$\mathcal{Y} = \begin{bmatrix} Z_1 & -Z_2 & -Z_3 & -Z_4 \\ Z_2 & Z_1 & -Z_4 & Z_3 \\ Z_3 & Z_4 & Z_1 & -Z_2 \\ Z_4 & -Z_3 & Z_2 & Z_1 \end{bmatrix},$$

where

$$Z_1 = \tfrac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}), \quad Z_2 = \tfrac{1}{4}(Y_{21} - Y_{12} + Y_{43} - Y_{34}),$$

$$Z_3 = \tfrac{1}{4}(Y_{31} - Y_{42} - Y_{13} + Y_{24}), \quad Z_4 = \tfrac{1}{4}(Y_{41} + Y_{32} - Y_{23} - Y_{14}).$$

Now, we construct a new quaternion (R, S)-symmetric matrix X using Z_1, Z_2, Z_3, and Z_4:

$$\begin{aligned} X &= Z_1 + Z_2 i + Z_3 j + Z_4 k = \tfrac{1}{4}\begin{bmatrix} I_m \\ iI_m \\ jI_m \\ kI_m \end{bmatrix}^T \mathcal{Y} \begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix} \\ &= \tfrac{1}{16}\begin{bmatrix} I_n \\ iI_m \\ jI_m \\ kI_m \end{bmatrix}^T \left(Y + Q_m Y Q_n{}^T + G_m Y G_n{}^T + T_m Y T_n{}^T\right) \begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix}. \end{aligned}$$

It is easy to verify that $X^\tau = \mathcal{Y}$. By (b) of Lemma 1, we obtain

$$(AXB)^\tau = A^\tau X^\tau B^\tau = A^\tau \mathcal{Y} B^\tau = C^\tau.$$

Thus, X satisfies (1). Moreover, $\mathcal{Y}$ is (R^τ, S^τ)-symmetric, and we have

$$(RXS)^\tau = R^\tau X^\tau S^\tau = R^\tau \mathcal{Y} S^\tau = \mathcal{Y} = X^\tau.$$

It implies that $RXS = X$, and so X is an (R, S)-symmetric solution to (1). Therefore, the consistency of (3) implies the consistency of (1). Moreover, any solution Y of (3) can generate an (R, S)-symmetric solution X of (1).

Next, we show that if (1) is consistent, then (3) is also consistent. Assume (1) has an (R, S)-symmetric solution X_0, i.e., $RX_0S = X_0$. By (b) of Lemma 1, we have

$$A^\tau X_0^\tau B^\tau = C^\tau$$

and

$$R^\tau X_0^\tau S^\tau = X_0^\tau.$$

Then we can see that $Y_0 = X_0^\tau$ is the (R^τ, S^τ)-symmetric solution of (3). Therefore, the original matrix Equation (1) is consistent if and only if its corresponding real matrix Equation (3) is consistent. According to Theorem 2.3 in [7], the real matrix Equation (3) has an (R^τ, S^τ)-symmetric solution if and only if the rank equalities in (c) of Theorem 1 hold. Additionally, when (3) is solvable, Y in the form of (5) is the general solution of the real matrix Equation (3), so we can generate our solution X by Equation (4). □

Similar to the proof of Theorem 1, we can obtain the following result for a skew symmetric case.

Theorem 2. *Let $A \in \mathbb{H}^{p\times m}$, $B \in \mathbb{H}^{n\times q}$, $C \in \mathbb{H}^{p\times q}$. Then there are three equivalent statements:*

(a) The matrix Equation (1) has an (R, S)-skew symmetric solution $X \in \mathbb{H}^{m\times n}$;

(b) The matrix equation

$$A^\tau Y B^\tau = C^\tau$$

has an (R^τ, S^τ)-skew symmetric solution $Y \in \mathbb{R}^{4m\times 4n}$;

(c) *The following rank equalities hold:*

$$r\begin{bmatrix} \mathcal{A}_2 & \mathcal{A}_1 & C^\tau \end{bmatrix} = r\begin{bmatrix} \mathcal{A}_2 & \mathcal{A}_1 \end{bmatrix}, r\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ C^\tau \end{bmatrix} = r\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix},$$

$$r\begin{bmatrix} C^\tau & \mathcal{A}_2 \\ \mathcal{B}_2 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & \mathcal{A}_2 \\ \mathcal{B}_2 & 0 \end{bmatrix}, r\begin{bmatrix} C^\tau & \mathcal{A}_1 \\ \mathcal{B}_1 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & \mathcal{A}_1 \\ \mathcal{B}_1 & 0 \end{bmatrix}.$$

Furthermore, if the matrix Equation (1) is solvable, then

$$X = \frac{1}{16}\begin{bmatrix} I_m \\ iI_m \\ jI_m \\ kI_m \end{bmatrix}^T \left(Y + Q_m Y Q_n{}^T + G_m Y G_n{}^T + T_m Y T_n{}^T\right)\begin{bmatrix} I_n \\ -iI_n \\ -jI_n \\ -kI_n \end{bmatrix}$$

is an (R, S)-skew symmetric solution to (1), where

$$Y = \begin{bmatrix} P & Q \end{bmatrix}\begin{bmatrix} 0 & Y_{PV} \\ Y_{QU} & 0 \end{bmatrix}\begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix}$$

with

$$Y_{PV} = \widetilde{Y}_{PV} + S_2 F_J U R_H T_2 + F_{\mathcal{A}_1} W_1 + W_2 E_{\mathcal{B}_2},$$

$$Y_{QU} = \tilde{Y}_{QU} + S_1 F_J U R_H T_1 + F_{\mathcal{A}_2} V_1 + V_2 E_{\mathcal{B}_1},$$

$$S_1 = \begin{bmatrix} I_{4m-r} & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & I_r \end{bmatrix}, \quad T_1 = \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 \\ I_{4n-k} \end{bmatrix}, \quad J = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}, \quad H = \begin{bmatrix} \mathcal{B}_1 \\ -\mathcal{B}_2 \end{bmatrix};$$

U, V_1, V_2, W_1, W_2 are arbitrary, and $\tilde{Y}_{PV}, \widetilde{Y}_{QU}$ are particular solutions to the matrix equation $\mathcal{A}_2 Y_{QU} \mathcal{B}_1 + \mathcal{A}_1 Y_{PV} \mathcal{B}_2 = C^\tau$.

3. Least-Squares (R, S)-(Skew) Symmetric Solutions

In this section, we derive the least-squares (R, S)-(skew) symmetric solutions to the quaternion matrix Equation (1). Let $M = M_1 + M_2 i + M_3 j + M_4 k \in \mathbb{H}^{p\times m}$, where $M_i \in \mathbb{R}^{p\times m}$. We define

$$M_c^\tau = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix}.$$

We can check that for any $N \in \mathbb{H}^{m\times q}$,

$$(MN)_c^\tau = M^\tau N_c{}^\tau. \tag{7}$$

To simplify the least-squares (R, S)-(skew) symmetric solution problems, we are going to use the Frobenius norm of $M_c{}^\tau$. By direct calculation, we first derive the relation between $\|M_c{}^\tau\|_F$ and $\|M||_F$:

$$\|M||_F = \frac{1}{2}\|M^\tau||_F = \|M_c{}^\tau\|_F. \tag{8}$$

Let $A \in \mathbb{H}^{p\times m}$, $B \in \mathbb{H}^{n\times q}$, $C \in \mathbb{H}^{p\times q}$. We will find an (R, S)-(skew) symmetric $X = \pm RXS \in \mathbb{H}^{m\times n}$ such that

$$\|AXB - C\|_F = \min_{X_0 \in \mathbb{H}^{m\times n}, X_0 = \pm RX_0 S} \|AX_0 B - C\|_F.$$

The lemma about the least-squares solutions is given as follows:

Lemma 2. [22] *The solutions of the least-squares problem of the complex matrix equation $AX = B$ are*

$$X = A^{\dagger}B + (I - A^{\dagger}A)Z,$$

in which Z is arbitrary.

Now we recall two important decompositions for an (R, S)-symmetric (reps. (R, S)-skew symmetric) matrix over quaternions.

Lemma 3. [11] *Let $R \in \mathbb{H}^{m\times m}, S \in \mathbb{H}^{n\times n}$ be nontrivial involutory matrices with the decomposition*

$$R = E^{-1}\begin{bmatrix} I_{r_1} & 0 \\ 0 & -I_{m-r_1} \end{bmatrix}E,\ S = F^{-1}\begin{bmatrix} I_{r_2} & 0 \\ 0 & -I_{n-r_2} \end{bmatrix}F.$$

Then

(a) $X \in \mathbb{H}^{m\times n}$ *is (R, S)-symmetric if and only if X can be expressed as*

$$X = E^{-1}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}F$$

where $X_1 \in \mathbb{H}^{r_1\times r_2}, X_2 \in \mathbb{H}^{(m-r_1)\times(n-r_2)}$.

(b) $Y \in \mathbb{H}^{m\times n}$ *is (R, S)-skew symmetric if and only if Y can be expressed as*

$$Y = E^{-1}\begin{bmatrix} 0 & Y_1 \\ Y_2 & 0 \end{bmatrix}F$$

where $Y_1 \in \mathbb{H}^{r_1\times(n-r_2)},\ Y_2 \in \mathbb{H}^{(m-r_1)\times r_2}$.

Next, we provide the main result of this section.

Theorem 3. *Let $A \in \mathbb{H}^{p\times m}$, $B \in \mathbb{H}^{n\times q}$, $C \in \mathbb{H}^{p\times q}$. Then each least-squares (R, S)-symmetric solution to the matrix equation $AXB = C$ should be in the form of*

$$X = E^{-1}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}F,$$

where E and F are given in Lemma 3,

$$Vec(X_1) = \begin{bmatrix} I_1 & 0_1 & iI_1 & 0_1 & jI_1 & 0_1 & kI_1 & 0_1 \end{bmatrix}y,$$
$$Vec(X_2) = \begin{bmatrix} 0_2 & I_2 & 0_2 & iI_2 & 0_2 & jI_2 & 0_2 & kI_2 \end{bmatrix}y,$$
$$y = P^{\dagger}b + (I - P^{\dagger}P)z,\ P = [{B_1}^T \otimes A_1, {B_2}^T \otimes A_2]^{\tau},\ b = (Vec(C))_c^{\tau},$$

0_1 is the zero matrix with the size of $r_1r_2 \times (m - r_1)(n - r_2)$,

0_2 is the zero matrix with the size of $(m - r_1)(n - r_2) \times r_1r_2$,

I_1 is the identity matrix with the size of $r_1r_2 \times r_1r_2$,

I_2 is the identity matrix with the size of $k \times k$,

where $k = (m - r_1)(n - r_2) \times (m - r_1)(n - r_2)$,

$z \in \mathbb{C}^{4r_1r_2+4(m-r_1)(n-r_2)}$ is arbitrary.

Proof. Using the decomposition of an (R,S)-symmetric matrix X in (a) in Lemma 3, we partition the matrices:

$$AE^{-1} = [A_1\ A_2], \quad FB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $A_1 \in \mathbb{H}^{p\times r_1}$, $A_2 \in \mathbb{H}^{p\times(m-r_1)}$, $B_1 \in \mathbb{H}^{r_2\times q}$, $B_2 \in \mathbb{H}^{(n-r_2)\times q}$. Then $AXB = C$ has a least-squares (R,S)-symmetric solution X if and only if $AE^{-1}\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}FB = C$; that is,

$$A_1X_1B_1 + A_2X_2B_2 = C$$

has a least-squares solution $\{X_1, X_2\}$. Using the vec operation, we have

$$({B_1}^T \otimes A_1)\mathrm{Vec}(X_1) + ({B_2}^T \otimes A_2)\mathrm{Vec}(X_2) = \mathrm{Vec}(C),$$

which can be rewritten as

$$[{B_1}^T \otimes A_1, {B_2}^T \otimes A_2]\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix} = \mathrm{Vec}(C). \tag{9}$$

Taking the real representations on both sides of (9),

$$[{B_1}^T \otimes A_1, {B_2}^T \otimes A_2]^\tau \left(\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix}\right)^\tau = \mathrm{Vec}(C)^\tau.$$

By Equations (7) and (8),

$$\begin{aligned} &\|[{B_1}^T \otimes A_1, {B_2}^T \otimes A_2]^\tau \left(\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix}\right)^\tau - (\mathrm{Vec}(C))^\tau\|_F \\ =&2\|[{B_1}^T \otimes A_1, {B_2}^T \otimes A_2]^\tau \left(\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix}\right)_c^\tau - (\mathrm{Vec}(C))_c^\tau\|_F. \end{aligned}$$

Now, we derive the least-squares solution to the linear system

$$Py = b,$$

where

$$P = [{B_1}^T \otimes A_1, {B_2}^T \otimes A_2]^\tau,\ y = \left(\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix}\right)_c^\tau,\ b = (\mathrm{Vec}(C))_c{}^\tau,$$

and Lemma 2 tells us that $y = P^\dagger b + (I - P^\dagger P)z$ is the least-squares solution. If we denote $\mathrm{Vec}(X_1) = \mathrm{Vec}(X_{11}) + \mathrm{Vec}(X_{21})i + \mathrm{Vec}(X_{31})j + \mathrm{Vec}(X_{41})k$ and $\mathrm{Vec}(X_2) = \mathrm{Vec}(X_{12}) + \mathrm{Vec}(X_{22})i + \mathrm{Vec}(X_{32})j + \mathrm{Vec}(X_{42})k$, then

$$\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix} = \begin{bmatrix} \mathrm{Vec}(X_{11}) \\ \mathrm{Vec}(X_{12}) \end{bmatrix} + \begin{bmatrix} \mathrm{Vec}(X_{21}) \\ \mathrm{Vec}(X_{22}) \end{bmatrix}i + \begin{bmatrix} \mathrm{Vec}(X_{31}) \\ \mathrm{Vec}(X_{32}) \end{bmatrix}j + \begin{bmatrix} \mathrm{Vec}(X_{41}) \\ \mathrm{Vec}(X_{42}) \end{bmatrix}k.$$

By the definition of $\left(\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix}\right)_c^\tau$, we have

$$y = \left(\begin{bmatrix} \mathrm{Vec}(X_1) \\ \mathrm{Vec}(X_2) \end{bmatrix}\right)_c^\tau = \begin{bmatrix} \mathrm{Vec}(X_{11}) \\ \mathrm{Vec}(X_{12}) \\ \mathrm{Vec}(X_{21}) \\ \mathrm{Vec}(X_{22}) \\ \mathrm{Vec}(X_{31}) \\ \mathrm{Vec}(X_{32}) \\ \mathrm{Vec}(X_{41}) \\ \mathrm{Vec}(X_{42}) \end{bmatrix}.$$

Clearly,

$$\mathrm{Vec}(X_1) = \begin{bmatrix} I_1 & 0_1 & iI_1 & 0_1 & jI_1 & 0_1 & kI_1 & 0_1 \end{bmatrix} y,$$
$$\mathrm{Vec}(X_2) = \begin{bmatrix} 0_2 & I_2 & 0_2 & iI_2 & 0_2 & jI_2 & 0_2 & kI_2 \end{bmatrix} y,$$

0_1 is the zero matrix with the size of $r_1 r_2 \times (m - r_1)(n - r_2)$, and 0_2 is the zero matrix with the size of $(m - r_1)(n - r_2) \times r_1 r_2$. Then $X = E^{-1} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} F$ is our required solutions. □

By using the decomposition in (b) of Lemma 3, we can obtain the general expression of the least-squares (R, S)-skew symmetric solution as follows.

Theorem 4. *Let $A \in \mathbb{H}^{p\times m}$, $B \in \mathbb{H}^{n\times q}$, $C \in \mathbb{H}^{p\times q}$. Then each least-squares (R, S)-skew symmetric solution to the matrix equation $AXB = C$ should be in the form of*

$$X = E^{-1} \begin{bmatrix} 0 & Y_1 \\ Y_2 & 0 \end{bmatrix} F,$$

where E and F are given in Lemma 3, and $\{A_1, A_2\}$ and $\{B_1, B_2\}$ are block matrices determined by AE^{-1} and FB, respectively.

$$Vec(Y_1) = \begin{bmatrix} I_3 & 0_3 & iI_3 & 0_3 & jI_3 & 0_3 & kI_3 & 0_3 \end{bmatrix} y,$$
$$Vec(Y_2) = \begin{bmatrix} 0_4 & I_4 & 0_4 & iI_4 & 0_4 & jI_4 & 0_4 & kI_4 \end{bmatrix} y,$$
$$y = P^{\dagger} b + (I - P^{\dagger} P) z,\ P = [{B_2}^T \otimes A_1, {B_1}^T \otimes A_2]^{\tau}, b = (Vec(C))_c{}^{\tau},$$

0_3 is the zero matrix with the size of $r_1(n - r_2) \times (m - r_1) r_2$,

0_4 is the zero matrix with the size of $(m - r_1) r_2 \times r_1(n - r_2)$,

I_3 is the identity matrix with the size of $r_1(n - r_2) \times r_1(n - r_2)$,

I_4 is the identity matrix with the size of $(m - r_1) r_2 \times (m - r_1) r_2$,

$z \in \mathbb{C}^{4(nr_1 + mr_2 - 2r_1 r_2)}$ is arbitrary.

4. Numerical Example

In this section, a numerical example is provided to illustrate our results. Here, we only consider the (R, S)-symmetric solution case.

Example 1. *For the given quaternion matrix equation,*

$$AXB = C,$$

where

$$A = \begin{bmatrix} 1 + 2i + 3j & 3i + j + 2k & 1 + i + 3k \\ 2 + 3j + k & j + 2k & 2i + j + 3k \end{bmatrix},$$
$$B = \begin{bmatrix} 2i - j + k & 2 + j + k \\ 3 + i - 5j & 4 + 2i - 3k \end{bmatrix},$$
$$C = \begin{bmatrix} 4 + 10.24i - 0.16j + 3.4k & 6.92 - 0.2i + 9.64j + 7.44k \\ -9.52 + 20.96i + 26.36j + 3.8k & 0.32 - 3.68i + 20.6j + 17.4k \end{bmatrix}.$$

Find the (R, S)-symmetric solutions if the matrix equation is solvable, where

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.28 & 0.96i \\ 0 & -0.96i & -0.28 \end{bmatrix},\ S = \begin{bmatrix} 0.28 & -0.96k \\ 0.96k & -0.28 \end{bmatrix}$$

are two involution matrices.

Step 1. By Theorem 1, consider the (R^τ, S^τ)*-symmetric solution to the corresponding real matrix equation*

$$A^\tau Y B^\tau = C^\tau, \tag{10}$$

where $A^\tau \in \mathbb{R}^{8\times 12}$, B^τ, $C^\tau \in \mathbb{R}^{8\times 8}$, *and* Y *is an unknown matrix.*

Step 2. Find P, Q, U, V, *such that* $R^\tau P = P$, $R^\tau Q = -Q$ *and* $S^\tau U = U$, $S^\tau V = -V$. *According to Trench [8],* P, Q *can be obtained from an orthonormal basis for the eigenspace of* R^τ *associated with* $\lambda = 1, -1$, *respectively. Similarly,* U, V *can be obtained from an orthonormal basis for the eigenspace of* S^τ *associated with* $\lambda = 1, -1$, *respectively.*

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -0.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.8 & 0 \\ 0 & 0 & 0 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 0 \\ -0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0 \\ 0 & -0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 \\ 0 & 0 & -0.8 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & -0.8 \end{bmatrix},$$

$$U = \begin{bmatrix} 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & -0.6 \\ 0.8 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0.6 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & 0 & 0 & 0.6 \\ -0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & -0.8 & 0 & 0 \\ 0 & -0.6 & 0 & 0 \\ 0 & 0 & -0.8 & 0 \\ -0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.8 \end{bmatrix},$$

and $\widehat{U}, \widehat{V}$ *are calculated from (2).*

Step 3. Perform the following decomposition and verify whether (10) has a solution or not.

$$A^\tau \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix}, \quad \mathcal{A}_1 \in \mathbb{R}^{8\times 8}, \mathcal{A}_2 \in \mathbb{R}^{8\times 4},$$

$$\begin{bmatrix} \widehat{U} \\ \widehat{V} \end{bmatrix} B^\tau = \begin{bmatrix} (\mathcal{B}_1)^T & (\mathcal{B}_2)^T \end{bmatrix}^T, \quad \mathcal{B}_1 \in \mathbb{R}^{4\times 8}, \mathcal{B}_2 \in \mathbb{R}^{4\times 8}.$$

By MATLAB, we have

$$r\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 & C^\tau \end{bmatrix} = r\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix} = 8, \quad r\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ C^\tau \end{bmatrix} = r\begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} = 8,$$

$$r\begin{bmatrix} C^\tau & \mathcal{A}_1 \\ \mathcal{B}_2 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & \mathcal{A}_1 \\ \mathcal{B}_2 & 0 \end{bmatrix} = 12, \quad r\begin{bmatrix} C^\tau & \mathcal{A}_2 \\ \mathcal{B}_1 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & \mathcal{A}_2 \\ \mathcal{B}_1 & 0 \end{bmatrix} = 8.$$

Thus, it follows from Theorem 1 that the matrix Equation (10) is consistent.

Step 4. Calculate the (R^{τ}, S^{τ})*-symmetric solution* Y *of the matrix Equation* (10) *by* (5)*:*

$$Y = \begin{bmatrix} -0.80 & 0 & 0 & -1.20 & 1.60 & 0 & 0 & -0.60 \\ -1.64 & 0.48 & 1.28 & 0 & 1.00 & 0.96 & 0.36 & -0.48 \\ -0.96 & 1.00 & -0.48 & -0.64 & -0.48 & -1.36 & 0 & -0.72 \\ 0 & 1.20 & -0.80 & 0 & 0 & -0.60 & -1.60 & 0 \\ -1.28 & 0 & -1.64 & 0.48 & 0.36 & -0.48 & -1.00 & -0.96 \\ 0.48 & 0.64 & -0.96 & 1.00 & 0 & -0.72 & 0.48 & 1.36 \\ -1.60 & 0 & 0 & 0.60 & -0.80 & 0 & 0 & -1.20 \\ -1.00 & -0.96 & -0.36 & 0.48 & -1.64 & 0.48 & 1.28 & 0 \\ 0.48 & 1.36 & 0 & 0.72 & -0.96 & 1.00 & -0.48 & -0.64 \\ 0 & 0.60 & 1.60 & 0 & 0 & 1.20 & -0.80 & 0 \\ -0.36 & 0.48 & 1.00 & 0.96 & -1.28 & 0 & -1.64 & 0.48 \\ 0 & 0.72 & -0.48 & -1.36 & 0.48 & 0.64 & -0.96 & 1.00 \end{bmatrix}.$$

Step 5. Generate X *from* Y *by the formula* (4)*:*

$$\begin{aligned} X &= \begin{bmatrix} -0.80 & 0 \\ -1.64 & 0.48 \\ -0.96 & 1.00 \end{bmatrix} + \begin{bmatrix} 0 & 1.20 \\ -1.28 & 0 \\ 0.48 & 0.64 \end{bmatrix} i \\ &+ \begin{bmatrix} -1.60 & 0 \\ -1.00 & -0.96 \\ 0.48 & 1.36 \end{bmatrix} j + \begin{bmatrix} 0 & 0.60 \\ -0.36 & 0.48 \\ 0 & 0.72 \end{bmatrix} k. \end{aligned}$$

Step 6. Verify whether X *is an* (R, S)*-symmetric solution to* $AXB = C$ *or not. By direct computation,* X *satisfies*

$$RXS = X, AXB = C.$$

Thus, X *is our required solution.*

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