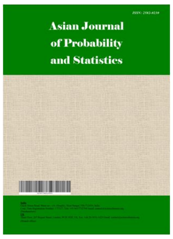

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The Zubair-Inverse Lomax Distribution with Applications

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this article, an extension of Inverse Lomax (IL) distribution with the Zubair-G family is considered . Various statistical properties of the new model where derived, including moment generating function, Rényi entropy, and order statistics. A Monte Carlo simulation study was presented to evaluate the performance of the maximum likelihood estimators. The new model can be skew to the right, constant, and decreasing functions depending on the parameter values. We discussed the estimation of the model parameters by maximum likelihood method. The application of the new model to the data sets indicates that the new model is better than the existing competitors as it has minimum value of statistics criteria.

Keywords: Zubair-G; Inverse Lomax; Simulation; Entropy; Monte Carlo.

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1 Introduction

The Inverse Lomax (IL) distribution is used in different fields, like stochastic modeling, Economics, actuarial sciences and lifestyle testing; it is one of the leading predictive life-time models. Inverse

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Lomax distribution is part of a Beta form distribution. Other family members include Singh maddala, Pareto, log-Logistics, Dagum, generalized second-type beta distributions among others as in [1]. Since then, IL distribution has gained a lot of coverage in many fields such as Actuarial Science and Economics (see [1]), Geophysical data (see [2]), Survival analysis (see [3]), and Medical Science (see [4] and [5]). Recently, [6] considered multicomponent stress-strength reliability for the IL distribution with different shape parameters.

The cdf and pdf of IL distribution are given by

$$G(x;\beta,\lambda)=\left(1+\frac{\lambda}{x}\right)^{-\beta} \qquad x>0,\beta,\lambda>0 \tag{1.1}$$

$$g(x;\beta,\lambda)=\beta\lambda x^{-2}\left(1+\frac{\lambda}{x}\right)^{-(1+\beta)} \qquad x>0,\beta,\lambda>0 \tag{1.2}$$

The motivations for this research are; to provide a better fits than the other models of IL distribution with the same number of parameters, and to improve the flexibility and characteristics of the IL distribution.

2 The Zubair-G Family

Recently, [7] introduced the Zubair-G family of distributions with the cumulative density function (cdf) and probability density function (pdf) given by

$$F_{ZG}(x;\alpha,\kappa)=\frac{\exp\{\alpha G(x;\kappa)^2\}-1}{\exp\{\alpha\}-1} \qquad x>0,\alpha>0 \tag{2.1}$$

$$f_{ZG}(x;\alpha,\kappa)=\frac{2\alpha g(x;\kappa)G(x;\kappa)\exp\{\alpha G(x;\kappa)^2\}}{\exp\{\alpha\}-1} \qquad x>0,\alpha>0 \tag{2.2}$$

Where κ is a vector of parameter/parameters for any baseline distribution.

3 The Zubair-Inverse Lomax Distribution

Based on Eqns. (2.1) and (2.2), we can insert Eqns. (1.1) and (1.2) and come up with the Zubair-Inverse Lomax (Z-IL) distribution. Below are the cdf, pdf, survival (S(x)), hazard (h(x)), reverse hazard (r(x)), and cumulative hazard (H(x)) functions of Z-IL distribution, respectively.

$$F_{Z-IL}(x;\alpha,\beta,\lambda)=\frac{\exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}-1}{\exp\{\alpha\}-1} \qquad x>0,\alpha,\beta,\lambda>0 \tag{3.1}$$

$$f_{Z-IL}(x;\alpha,\beta,\lambda)=\frac{2\alpha\beta\lambda x^{-2}\left(1+\frac{\lambda}{x}\right)^{-(1+2\beta)}\exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}}{\exp\{\alpha\}-1} \qquad x>0,\alpha,\beta,\lambda>0 \tag{3.2}$$

$$S_{Z-IL}(x;\alpha,\beta,\lambda)=\frac{\exp\{\alpha\}-\exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}}{\exp\{\alpha\}-1} \qquad x>0,\alpha,\beta,\lambda>0 \tag{3.3}$$

$$h_{Z-IL}(x;\alpha,\beta,\lambda)=\frac{2\alpha\beta\lambda x^{-2}\left(1+\frac{\lambda}{x}\right)^{-(1+2\beta)}\exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}}{\exp\{\alpha\}-\exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}} \qquad x>0,\alpha,\beta,\lambda>0 \tag{3.4}$$

$$r_{Z-IL}(x;\alpha,\beta,\lambda) = \frac{2\alpha\beta\lambda x^{-2}\left(1+\frac{\lambda}{x}\right)^{-(1+2\beta)}\exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}}{\exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}-1} \qquad x>0,\alpha,\beta,\lambda>0 \tag{3.5}$$

and

$$H_{Z-IL}(x;\alpha,\beta,\lambda) = -log\left[\frac{\exp\{\alpha\} - \exp\{\alpha\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}}{\exp\{\alpha\}-1}\right] \tag{3.6}$$

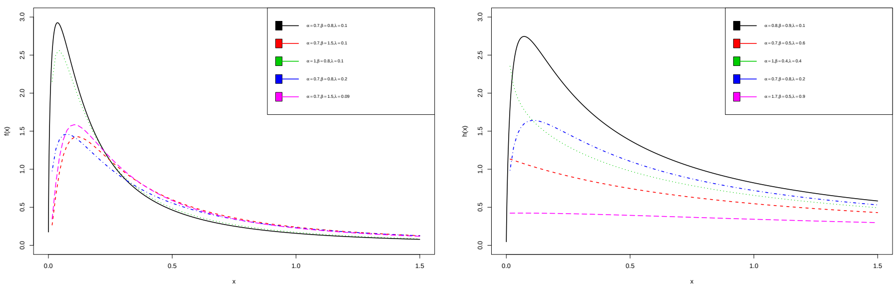

Fig. 1. The pdf and hazard function plots of Z-IL distribution with various parameter values

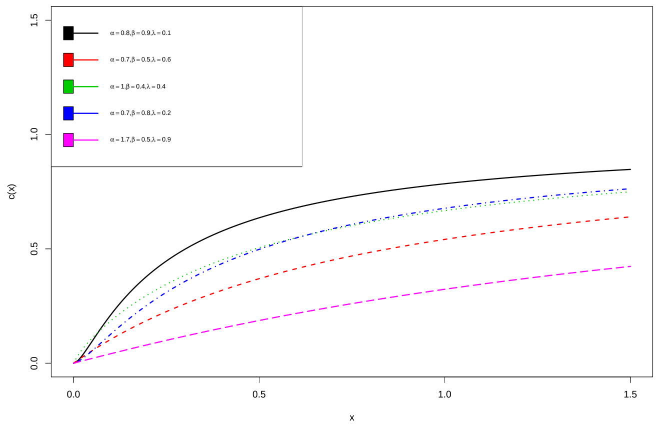

Fig. 2. The cdf plot of Z-IL distribution with various parameter values

4 Statistical Properties of Zubair-Inverse Lomax (Z-IL) Distribution

In this section, we considered some of the statistical properties of the Z-IL distribution like moments, mgf, Rényi entrophy, and order statistics.

4.1 Moments

Suppose X is a random variable with density function defined in Eqn. (3.2), the r^{th} non-central moments of X is given by

$$\begin{aligned}E(X^r) &= \int_{-\infty}^{\infty} x^r f(x)dx \\ &= 2\sum_{i=0}^{\infty} \frac{\alpha^{i+1}\beta\lambda}{(\exp\{\alpha\}-1)i!}\int_0^{\infty} x^{r-2}\left(1+\frac{\lambda}{x}\right)^{-[2\beta(1+i)+1]} dx\end{aligned}$$

by letting $y = \frac{\lambda}{x}$ and some simplifications, we have

$$\begin{aligned}E(X^r) &= 2\sum_{i=0}^{\infty} \frac{\alpha^{i+1}\beta\lambda^r}{(\exp\{\alpha\}-1)i!}\int_0^{\infty} y^{-r}\left(1+\frac{\lambda}{x}\right)^{-[2\beta(1+i)+1]} dy \\ &= 2\sum_{i=0}^{\infty} \frac{\alpha^{i+1}\beta\lambda^r}{(\exp\{\alpha\}-1)i!} B((1-r),(2\beta(1+i)+r)) \qquad (4.1)\end{aligned}$$

where $\int_0^{\infty} \frac{t^{a-1}}{(1+t)^{a+b}}dt = B(a,b)$ is Beta function of second kind.

4.2 Moment generating function

The moment generating function (mgf) of the Z-II distribution ca be given interms of 4.1 as

$$M_x(x) = 2\sum_{i,r=0}^{\infty} \frac{t^r\alpha^{i+1}\beta\lambda^r}{(\exp\{\alpha\}-1)i!r!} B((1-r),(2\beta(1+i)+r)) \qquad (4.2)$$

4.3 Rényi entrophy

If X is random variable with density function $f(x)$ defined in Eqn. (3.2), then the Rényi entropy of X is given by

$$R_\tau(x) = \frac{1}{1-\tau}\left[\int_{-\infty}^{\infty} f(x)^\tau dx\right], \qquad \tau > 0, \tau \neq 1; x \in \Re \qquad (4.3)$$

the function $f(x)^\tau$ in Eqn. (4.3) can be presented as

$$f(x)^\tau = \left[\frac{2\alpha\beta\lambda}{(\exp\{\alpha\}-1)}\right]^\tau x^{-2\tau}\left(1+\frac{\lambda}{x}\right)^{-\tau(1+2\beta)} \exp\{\alpha\tau\left(1+\frac{\lambda}{x}\right)^{-2\beta}\} \qquad (4.4)$$

By inserting Eqn. (4.4) back in Eqn. (4.3), we have

$$\int_{-\infty}^{\infty} f(x)^\tau dx = \left[\frac{2\alpha\beta\lambda}{(\exp\{\alpha\}-1)}\right]^\tau \int_0^{\infty} x^{-2\tau}\left(1+\frac{\lambda}{x}\right)^{-\tau(1+2\beta)} \exp\{\alpha\tau\left(1+\frac{\lambda}{x}\right)^{-2\beta}\}dx \qquad (4.5)$$

$$= 2^\tau \sum_{i=0}^{\infty} \frac{t^i\alpha^{i+\tau}(\beta\lambda)^\tau}{(\exp\{\alpha\}-1)^\tau i!}\int_0^{\infty} x^{-2\tau}\left(1+\frac{\lambda}{x}\right)^{-[\tau(1+2\beta)+2\beta i]} dx \qquad (4.6)$$

Let $w = \frac{\lambda}{x}$, after some simplifications we have

$$\int_{-\infty}^{\infty} f(x)^\tau dx = 2^\tau \sum_{i=0}^{\infty} \frac{t^i\alpha^{i+\tau}\beta^\tau\lambda^{1-\tau}}{(\exp\{\alpha\}-1)^\tau i!}\int_0^{\infty} \frac{w^{2\tau-1-1}}{(1+w)^{[\tau(2\beta+1)+2\beta i]}} dw \qquad (4.7)$$

Finnaly, the Rényi entropy is given by

$$R_\tau(x) = 2^\tau \sum_{i=0}^{\infty} \frac{t^i \alpha^{i+\tau} \beta^\tau \lambda^{1-\tau}}{(1-\tau)(\exp\{\alpha\}-1)^\tau i!} B(2\tau - 1, \frac{\tau}{\beta} + 2i + 1) \tag{4.8}$$

4.4 Order statistic

Let $X_1, X_2, X_3, X_4 \dots X_n$ be the random samples of size n from probability distribution with pdf $f(x)$ and cdf $F(x)$ as defined in Eqns. (3.2) and (3.1), respectively. Suppose $X_{1:n}, X_{2:n}, X_{3:n}, X_{4:n} \dots X_{n:n}$ denoted the corresponding order statistics derived from this samples. Then, the p^{th} order statistic can be defined as

$$f_{p:n}(x) = \frac{n! f(x)}{(p-1)!(n-p)!} F(x)^{p-1} \left[1 - F(x)\right]^{n-p} \tag{4.9}$$

Eqn. (4.9) can be written as

$$\begin{aligned} f_{p:n}(x) = & \frac{2\alpha\beta\lambda n!}{(\exp\{\alpha\}-1)(p-1)!(n-p)!} x^{-2} \left(1 + \frac{\lambda}{x}\right)^{-(1+2\beta)} exp\{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}\} \\ & \left[\frac{\exp\{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}\} - 1}{\exp\{\alpha\} - 1}\right]^{i-1} \sum_{j=0}^{n-p} (-1)^j \binom{n-p}{j} \left[\frac{\exp\{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}\} - 1}{\exp\{\alpha\} - 1}\right]^j \end{aligned} \tag{4.10}$$

Therefore, the order statistics can be given by

$$f_{p:n}(x) = \Omega_j x^{-2} \left(1 + \frac{\lambda}{x}\right)^{-(1+2\beta)} exp\{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}\} \left[\exp\{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}\} - 1\right]^{p+j-1} \tag{4.11}$$

where $\Omega_j = 2 \sum_{j=0}^{\infty} \frac{\alpha\beta\lambda n! (-1)^j}{(\exp\{\alpha\}-1)^{p+j} (p-1)!(n-p)!} \binom{n-p}{j}$

5 Estimation

In this section, the parameters of the proposed Z-IL distribution will be estimated using maximum likelihood method. Let $x_1, x_2, x_3, \dots, x_n$ be random samples of n observations drawn from the Z-IL distribution with parameter vector $\eta = (\alpha, \beta, \lambda)^T$. The log-likelihood function of Eqn. (3.2) denoted by $L(\eta)$ can be written as

$$\begin{aligned} L(\eta) = & n \log(2\alpha\beta\lambda) - 2\sum_{i=1}^{n} log(x) - (1+2\beta) \sum_{i=1}^{n} log\left(1 + \frac{\lambda}{x}\right) \\ & + \alpha \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x}\right)^{-2\beta} - nlog\left(\exp\{\alpha\} - 1\right) \end{aligned} \tag{5.1}$$

By taking partial derivatives of Eqn. (5.1) with respect to α, β, and λ, we derived the components of score vector $U(\eta)$ presented as follows

$$U_\alpha(\eta) = \frac{n}{\alpha} + \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta} - \frac{n \exp\{\alpha\}}{(\exp\{\alpha\} - 1)} \tag{5.2}$$

$$U_\beta(\eta) = \frac{n}{\beta} - 2\sum_{i=1}^{n} log\left(1 + \frac{\lambda}{x_i}\right) + \alpha \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta} log\left(\alpha \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)\right) \tag{5.3}$$

$$U_\lambda(\eta) = \frac{n}{\lambda} + \sum_{i=1}^{n} \frac{(1+2\beta)}{(x_i + \frac{\lambda}{x_i^2})} - 2\alpha\beta \sum_{i=1}^{n} \frac{\left(1+\frac{\lambda}{x}\right)^{-2\beta-1}}{x_i} \tag{5.4}$$

By Setting Eqns. (5.2), (5.3), and (5.4) to zero and also solving them simultaneously yields the maximum likelihood estimators of the Z-IL distribution. However, the above equations are nonlinear they cannot be solved analytically. As such, the statistical software can be used to solve them numerically using numerical algorithm.

6 Monte Carlo Simulation

Here, a simulation study is conducted and presented to show the estimates' performance at various parameter values. Monte Carlo method is any computational technique using pseudo-random numbers to solve mathematical problems as defined by [8]. The numerical study is as follows:
1. For known parameter values i.e $\eta = (\alpha, \beta, \lambda)^T$, we simulated a random sample of size n from the Z-IL distribution using Eqt. (6.2).
2. We then Estimate the parameters of the Z-IL distribution by using MLE.
3. Perform 10,000 replications of steps 1 through 2.
4. For each of the three (3) parameters of the Z-IL, we compute the Bias, MSE, Variance, and Estimates of the parameters from the 10,000 parameter estimates. We considered two (2) cases; case 1 ($\alpha = .6$, $\beta = .4$, and $\lambda = .5$) whereas in case 2 ($\alpha = 1$, $\beta = .4$, and $\lambda = .5$). The statistics are given by

$$\hat{\eta} = \frac{1}{10,000} \sum_{i=1}^{10,000} \hat{\eta_i}, \qquad Bias(\hat{\eta}) = \hat{\eta_i} - \eta,$$

$$var(\hat{\eta}) = \sum_{i=1}^{10,000} \frac{(\hat{\eta_i} - \hat{\eta})^2}{10,000}$$

$$MSE(\hat{\eta}) = var(\hat{\eta}) + (Bias(\hat{\eta}))^2 \tag{6.1}$$

where $\hat{\eta_i} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ are the mle for each iteration ($n = 30, 50, 75, 125, 200, 300, 500, 600, 700$).The quantile function for Z-IL is giving by

$$Q_{Z-IL}(u) = \frac{\lambda}{\left[\left[\frac{log\{U(\exp\{\alpha\}-1)+1\}}{\alpha}\right]^{-\frac{1}{2\beta}} - 1\right]} \tag{6.2}$$

The numerical results are presented in Table 1. The simulation study has shown that irrespective of the parameter values chosen, the Bias and MSE of the parameter estimates (for both cases) decreases as the sample size n increases. Thus, the larger the sample size, the more accurate are the estimates of the parameters. The estimates are good in both cases, as they approaches the true parameter values as the sample sizes increases.

7 Application

We illustrate the applicability of the Z-IL distribution to three (3) data sets. The first data set repesents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports as in [9], and [10]. The summary of the data set includes: Minimum = 28, Maximum = 200, Mean = 69.02, Median = 58.60, Skewness = 1.1747, and Kurtosis = 4.3651. The second data set repesents the airborne communication transceiver as reported by [11]. The summary of the data set includes:

Table 1. A simulation results of Z-IL distribution for the 2 cases

		Case 1			Case 2		
n	properties	α=.6	β=.4	λ=.5	α=1	β=.4	λ=.5
30	Bias	0.2689	0.2796	0.056	-0.0633	0.3158	0.1286
	MSE	1.5943	2.961	0.2056	1.6031	2.9696	0.2818
	Est.	0.8689	0.6796	0.556	0.9367	0.7158	0.6286
50	Bias	0.3823	0.0607	0.0379	0.0085	0.0778	0.1159
	MSE	2.0148	0.4094	0.1174	1.8928	0.4095	0.1635
	Est.	0.9822	0.4608	0.5379	1.0085	0.4778	0.6159
75	Bias	0.4536	0.0046	0.0276	0.1555	0.0185	0.0984
	MSE	2.2726	0.0295	0.0827	2.225	0.0462	0.1182
	Est.	1.0536	0.4046	0.5276	1.555	0.4185	0.5984
125	Bias	0.4087	-0.012	0.0237	0.1312	-0.0029	0.0918
	MSE	2.2569	0.0125	0.0539	2.2589	0.0147	0.0796
	Est.	1.0086	0.3879	0.5237	1.1312	0.3971	0.5918
200	Bias	0.3966	-0.0186	0.0181	0.1277	-0.0106	0.0835
	MSE	2.0441	0.0082	0.0423	2.0976	0.0097	0.0611
	Est.	0.9966	0.3814	0.5181	1.1277	0.3894	0.5835
300	Bias	0.3586	0.0197	0.0145	0.1489	-0.0149	0.0734
	MSE	1.7518	0.0058	0.0358	1.8839	0.0068	0.0539
	Est.	0.9588	0.3803	0.5145	1.1489	0.385	0.5734
500	Bias	0.2349	-0.0155	0.0161	0.0544	-0.0017	0.06834
	MSE	1.1479	0.0035	0.0279	1.3446	0.0043	0.0431
	Est.	0.8349	0.3845	0.5161	1.0544	0.3883	0.5684
600	Bias	0.1921	0.0174	-0.0136	0.0199	-0.0098	0.0659
	MSE	0.9771	0.0254	0.0029	1.152	0.0036	0.0391
	Est.	0.7921	0.5174	0.3864	1.0199	0.3902	0.5659
700	Bias	0.1749	-0.1284	0.0165	0.0093	-0.0094	0.0632
	MSE	0.8399	0.0024	0.0244	1.0134	0.003	0.0373
	Est.	0.7749	0.3872	0.5165	1.0093	0.3906	0.5632

Minimum = .5, Maximum = 24.5, Mean = 4.013, Median = 2.1, Skewness = 2.717, and Kurtosis = 10.543. The Third data set repesents the stress-rupture life of kevlar 49/epoxy strands that are subjected to constant sustained pressure at the 90 per cent stress level until all have failed as reported by [12], [13], and [14]. The summary of the data set includes: Minimum = 1, Maximum = 789, Mean = 102.5, Median = 8, Skewness = 3, and Kurtosis = 16.71. The plots of the three data sets are in Figs. (3), (4), and (5), respectively. The data sets are presented in the Appendix B.

We used AdequacyModel package by [15] in R developed by [16]. The goodness of fit statistic used in comparing the performances includes Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent akaike information criterion (CAIC), and Hannan Quinn information Criteria (HQIC). The smaller the value of the goodness of fit measures the better the fit to the data.

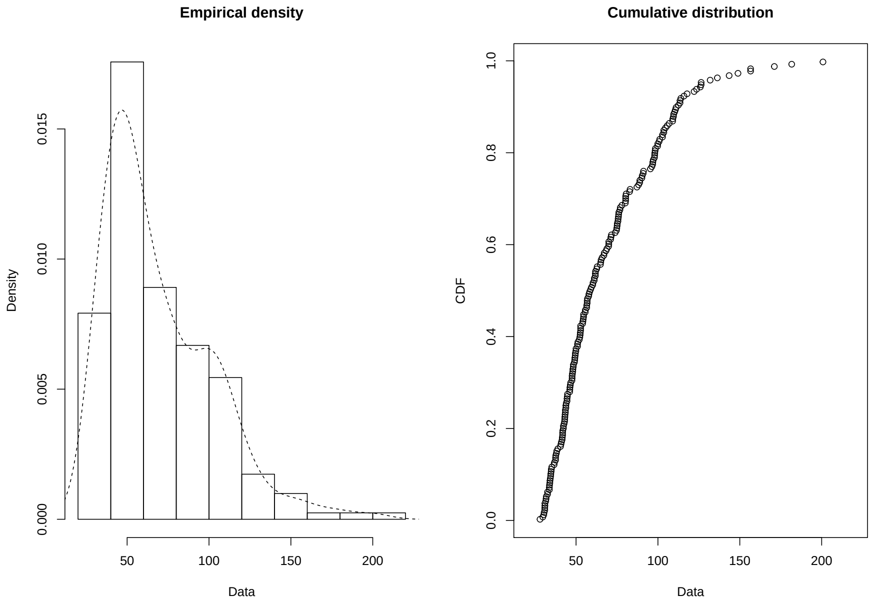

Fig. 3. The Density and CDF plots for the Sum of Skins Data Set

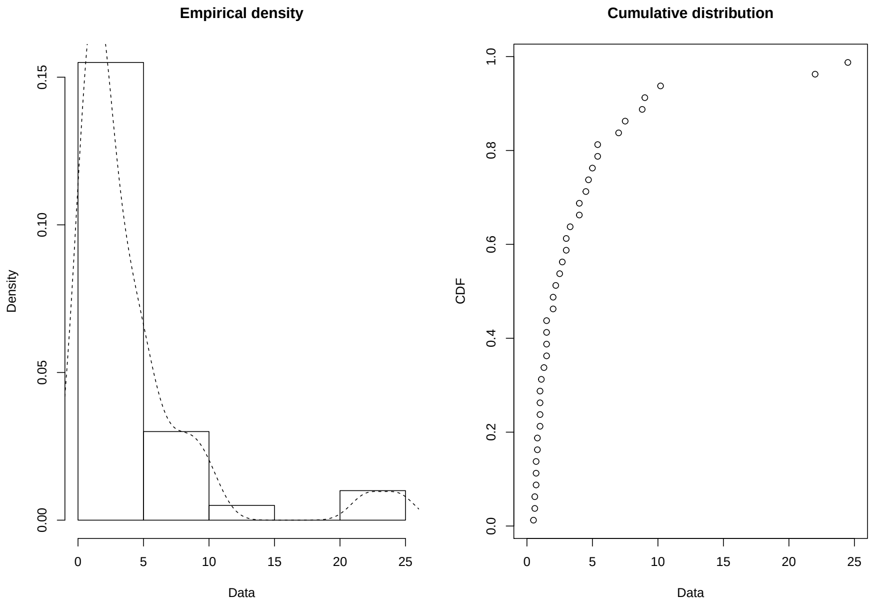

Fig. 4. The Density and CDF plots for the Airborne Data Set

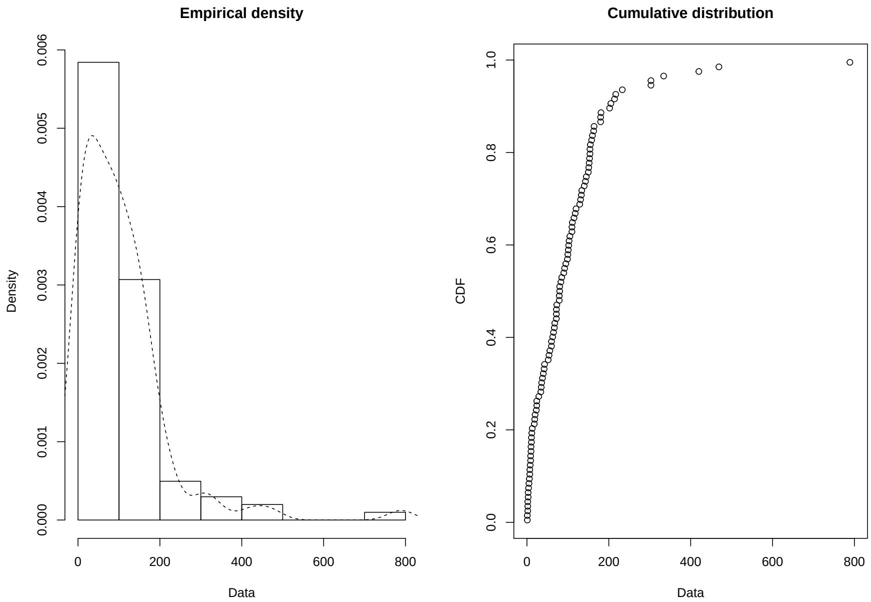

Fig. 5. The Density and CDF plots for the Failure Time Data Set

Table 2. Competing Models with Z-IL distribution

Models	Reference
IPL	[17]
APTIL	[18]
MOIL	[19]

Table 3. The MLEs, statistics, and -log likelihood for the Z-IL distribution with its competitors

		Estimates				Statistics				
Data Sets	Models	α	β	θ	λ	AIC	BIC	CAIC	HQIC	-ll
Data Set1	Z-IL	0.1232	0.1893		0.1893	4388.526	4398.451	4388.647	4392.541	2191.263
	IPL	0.1129	0.1058		0.1714	4422.604	4432.528	4422.725	4426.619	2208.302
	APTIL	0.1806	0.1988	0.1981		5048.348	5058.273	5048.47	5052.364	2521.174
	MOIL	0.1977	0.1977	0.1925		5323.575	5333.499	5323.696	5327.59	2658.787
	IL		0.1981		0.198	4654.587	4661.204	4654.647	4657.264	2325.294
Data Set2	Z-IL	0.1812	0.1992		0.1997	351.0375	356.1041	351.7041	352.8694	172.5187
	IPL	0.1976	0.1976		0.1643	376.5536	381.6202	377.2202	378.3855	185.2768
	APTIL	0.1769	0.1819	0.1773		503.1921	508.2587	503.8502	505.024	248.596
	MOIL	0.1695	0.1695	0.1778		572.3807	577.4473	573.0474	574.2126	283.1903
	IL		0.1987	0.1987		409.654	413.0318	409.9783	410.8753	202.827
Data Set3	Z-IL	0.1602	0.1828		0.1807	2140.868	2148.713	2141.115	2144.044	1067.434
	IPL	0.0113	0.1058		0.0171	2178.36	2186.205	2178.607	2181.536	1086.18
	APTIL	0.1933	0.1883	0.1947		2462.68	2470.525	2462.927	2465.856	1228.34
	MOIL	0.1977	0.1977	0.1925		2594.969	2602.814	2595.216	2598.145	1294.484
	IL		0.1981		0.198	2259.795	2265.025	2259.917	2261.912	1127.897

The competing models for the Z-IL model are presented in Table (2). They are the Inverse power lomax (IPL), the Aplpha Power Transformed Inverse Lomax (APTIL), and the Marshall-Olkin Inverse Lomax. As shown in Table (3), the Z-IL model is the best with a minimum values of all the statistics. Its ranked number 1 outperforming the other models.

8 Conclusion

In this paper, a new model based on Zubair-G family called the Zubair-Inverse Lomax (Z-IL) distribution was proposed and studied. Some of its statistical properties include Moments, Moment generating function, entropy and order statistics was investigated. The parameters of ZIL distribution were estimated using maximum likelihood method. The pdf plots in Fig. (1) indicates that the shape can be skew to the right whereas, the hazard function plots explains the shape as constant, skew to the right, and decreasing. Moreover, the cdf converges to one as in Fig. (2). An example of real world data sets empirically shows the importance and value of the new model.

Competing Interests

Authors have declared that no competing interests exist.

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Appendices

```
#### Appendix A #####

#### Plots #####

#################  PDF plot of Z-IL DISTRIBUTION ##############
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,1.5,0.0001)
y=function(x,alpha=.7,beta=.8,lambda=.1) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-1) plot(x,y(x),"l",col=1,lwd=2,ylab="f(x)",lty=1,ylim=c(0,3))


y=function(x,alpha=.7,beta=1.5,lambda=.1) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-1) curve(y,add=T,col=2,lwd=2,lty=2)

y=function(x,alpha=1,beta=.8,lambda=.1) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-1) curve(y,add=T,col=3,lwd=2,lty=3)
```

```
y=function(x,alpha=.7,beta=.8,lambda=.2) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-1)curve(y,add=T,col=4,lwd=2,lty=4)

y=function(x,alpha=.7,beta=1.5,lambda=.09) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-1)curve(y,add=T,col=6,lwd=2,lty=5)

legend("topright",legend=c(
expression(paste(alpha==.7,",",beta==.8,",",lambda==.1)),
expression(paste(alpha==.7,",",beta==1.5,",",lambda==.1)),
expression(paste(alpha==1,",",beta==.8,",",lambda==.1)),
expression(paste(alpha==.7,",",beta==.8,",",lambda==.2)),
expression(paste(alpha==.7,",",beta==1.5,",",lambda==.09))),
lwd=2,col=c(1,2,3,4,6),text.width=.5,cex=0.7,fill=c(1,2,3,4,6))
################################
################## Hazard Plot of Z-IL DISTRIBUTION ###############
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,1.5,0.0001)
y=function(x,alpha=.8,beta=.9,lambda=.1) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-exp(alpha*(1+(lambda/x))^(-2*beta)))
plot(x,y(x),"l",col=1,lwd=2,ylab="h(x)",lty=1,ylim=c(0,3))


y=function(x,alpha=.7,beta=.5,lambda=.6) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-exp(alpha*(1+(lambda/x))^(-2*beta)))
curve(y,add=T,col=2,lwd=2,lty=2)

y=function(x,alpha=1,beta=.4,lambda=.4) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-exp(alpha*(1+(lambda/x))^(-2*beta)))
curve(y,add=T,col=3,lwd=2,lty=3)

y=function(x,alpha=.7,beta=.8,lambda=.2) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-exp(alpha*(1+(lambda/x))^(-2*beta)))
curve(y,add=T,col=4,lwd=2,lty=4)

y=function(x,alpha=1.7,beta=.5,lambda=.9) (2*alpha*beta*lambda*x^(-2)
*(1+(lambda/x))^(-1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/
(exp(alpha)-exp(alpha*(1+(lambda/x))^(-2*beta)))
curve(y,add=T,col=6,lwd=2,lty=5)

legend("topright",legend=c(
expression(paste(alpha==.8,",",beta==.9,",",lambda==.1)),
expression(paste(alpha==.7,",",beta==.5,",",lambda==.6)),
expression(paste(alpha==1,",",beta==.4,",",lambda==.4)),
expression(paste(alpha==.7,",",beta==.8,",",lambda==.2)),
```

```
expression(paste(alpha==1.7,",",beta==.5,",",lambda==.9))),
lwd=2,col=c(1,2,3,4,6), text.width=.5,cex=0.7,fill=c(1,2,3,4,6))
############## Cdf plot of Z-IL #################
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,1.5,0.0001)
y=function(x,alpha=.8,beta=.9,lambda=.1) (exp(alpha
*(1+(lambda/x))^(-2*beta))-1)/(exp(alpha)-1)
plot(x,y(x),"l",col=1,lwd=2,ylab="c(x)",lty=1,ylim=c(0,1.5))


y=function(x,alpha=.7,beta=.5,lambda=.6) (exp(alpha
*(1+(lambda/x))^(-2*beta))-1)/(exp(alpha)-1)
curve(y,add=T,col=2,lwd=2,lty=2)

y=function(x,alpha=1,beta=.4,lambda=.4) (exp(alpha
*(1+(lambda/x))^(-2*beta))-1)/(exp(alpha)-1)
curve(y,add=T,col=3,lwd=2,lty=3)

y=function(x,alpha=.7,beta=.8,lambda=.2) (exp(alpha
*(1+(lambda/x))^(-2*beta))-1)/(exp(alpha)-1)
curve(y,add=T,col=4,lwd=2,lty=4)

y=function(x,alpha=1.7,beta=.5,lambda=.9) (exp(alpha
*(1+(lambda/x))^(-2*beta))-1)/(exp(alpha)-1)
curve(y,add=T,col=6,lwd=2,lty=5)

legend("topleft",legend=c(
expression(paste(alpha==.8,",",beta==.9,",",lambda==.1)),
expression(paste(alpha==.7,",",beta==.5,",",lambda==.6)),
expression(paste(alpha==1,",",beta==.4,",",lambda==.4)),
expression(paste(alpha==.7,",",beta==.8,",",lambda==.2)),
expression(paste(alpha==1.7,",",beta==.5,",",lambda==.9))),
lwd=2,col=c(1,2,3,4,6), text.width=.5,cex=0.7,fill=c(1,2,3,4,6))
#
#################################

#### Appendix B ######

### Data Sets #####
#airborne communication transceiver data.
x=c(0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00,
1.00, 1.10, 1.30, 1.50,
1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30,
 4.00, 4.00, 4.50, 4.70,5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00,
10.20, 22.00, 24.50)

#failure times in hours stress-rupture life of kevlar 49/epoxy strands
x=c(001,001,002,002,002,003,003,004,005,006,007,007,008,009,009,010,010,
011,011,012,013, 018,019,020,023,
024,024,029,034,035,036,038,040,042,043,052,054,056,060,060,063,065,067,
```

```
068,072,072,072, 073,079,079,080,080,083,085,090,092,095,099,100,101,102,
103,105,110,110,111,115,118, 120,129,131,133,134,140,143,145,150,151,152,
153,154,154,155,158,160,163,164, 180,180,181,202,205,214,217,233,303,303,
334,420,469,789)
#sum of skins
x<-c(28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3,
109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8,
80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0,
65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3,
 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8,
47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9,
 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2,101.1, 47.5, 46.2, 38.2, 49.2,
49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6,
 73.9, 43.5, 61.8, 88.9, 31.0, 37.6,52.8, 97.9, 111.1, 114.0, 62.9, 36.8,
56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8,
61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6,
48, 61.2, 171.1, 113.5, 148.9, 49.9,59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6,
76.8,  99.8, 80.1, 57.9, 48.4, 41.8, 44.5,   43.8, 33.7, 30.9, 43.3, 117.8,
 80.3, 156.6, 109.6,    50.0, 33.7,  54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5,
 115.9,  98.5, 54.6, 50.9,    44.7, 41.8, 38.0, 43.2,70.0, 97.2, 123.6, 181.7,
136.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6,
46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3,
54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9)
```

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