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Existence and Stability of Equilibrium Points under Combined Effects of Oblateness and Triaxiality in the Restricted Problem of Four Bodies

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The restricted four-body problem consists of an infinitesimal body which is moving under the Newtonian gravitational attraction of three finite bodies m_1, m_2, m_3. The three bodies (primaries) lie always at the vertices of an equilateral triangle, while each moves in circle about the centre of mass of the system fixed at the origin of the coordinate system. The fourth body does not affect the motion of the three bodies. We consider that the dominant primary body m_1 and smaller primary m_2 are respectively triaxial and oblate spheroidal bodies. We investigate the existence and locations of the equilibrium points and study their linear stability for the case of two equal masses. The result shows that the non-sphericity of the bodies plays an important role on the existence and evolution of the equilibrium points and influences in a very definitive way their position, as well as, their stability.

Keywords: Restricted four-body problem; triaxiality; oblateness; equilibrium points; stability.

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1 Introduction

In celestial mechanics, one amidst various inspiring subject is the restricted three-body problem (R3BP). The R3BP consists of two massive bodies (primaries) moving in orbits (circular or elliptic) around their common barycenter and a third body of negligible mass being influenced, but not influencing them. The solution to this type of problem which has been developed over the centuries [1-5] form the basis of the study of the dynamics of celestial bodies, from the computation of the ephemerides to the recent advances in flight dynamics. Restricted four-body problem (R4BP) is a modification of the R3BP and a natural generalization of it. It describes the motion of a body of infinitesimal mass under the Newtonian gravitational attraction of three finite bodies, called primaries, whose trajectory are the solution of the three Newtonian body problems. In recent years, several modifications of this classical problem have been introduced in order to make it more relevant and applicable to certain systems of Dynamical Astronomy. In the present paper, we will consider the combination of two such modifications in which the variation of the triaxial factor and the oblateness of the primaries are considered.

The formulation of the classical R3BP considered all the bodies to be strictly spherical. But in actual sense, most celestial bodies are not perfect spheres. Some have the shape of an oblate spheroid while some are triaxial in nature. For example, the Earth, Jupiter, Saturn and stars (Archerner, Antares and Altair) have the shape of an oblate spheroid while the Moon and several Post Asymptotic Giant Branch stars (Post AGB), Haumea (a scalene dwarf planet), are triaxial in shape. The lack of sphericity of the planets causes large perturbations from a two-body orbit. This inspires many scientists, among others, [6-10] and references therein, to include the shapes of the bodies in their study of R3BP.

In the same vein, the classical R4BP may be generalized to include different types of effects such as radiation pressure force, triaxiality and oblateness of the bodies, Poynting-Robertson drag, Coriolis and centrifugal forces, etc.

In the general problem of three bodies there is a particular solution in which the bodies lie at the vertices of an equilateral triangle, each moving in a Keplerian orbit. This is well known, and was first studied by [1]. He found a solution where the three bodies remain at constant distances from each other while they revolved around their common center of mass. There has been recently an increased interest for this model (Lagrangian configuration) because of some observational evidence; as it is known the Sun, Jupiter and the Trojan asteroids formed such a configuration in our Solar system.

In this paper, we study the R4BP in which the primaries are in the Lagrange equilateral triangle configuration (see Section 2). In the framework of this model, many works have been done in the last years. For example, [11] studied the equilateral R4BP in the case where the three primaries have equal masses. [12] investigated the number of the equilibrium points of the problem for any value of the masses, and studied numerically their linear stability varying the values of the masses. Besides, they showed the regions of the basins of attraction for the equilibrium points for some values of the mass parameters. In [13], the authors studied the periodic orbits of the problem for the case of two equal masses approximately at Routh's critical value. Recently, [14] have examined the equilibrium points in the photogravitational R4BP for the case of two equal masses. Also, the linear stability of each equilibrium point was examined. [15] also studied the periodic solutions in the photogravitational case of the problem; [16] studied the stability regions of the equilibrium points of the problem by taking into account the oblateness of the two small primaries. They established eight equilibrium points, two collinear and six non-collinear points and further observed that the stability regions of the equilibrium points expanded due to the presence of oblateness coefficients and various values of Jacobi constant C. In the last year, [17] investigated the out-of-plane equilibrium points in the photogravitational R4BP; however, they considered all the primary bodies as radiation sources with two of the bodies having the same radiation and mass value. More recently, [18] extended the work of [17] by considering the shape of the smaller primaries as oblate spheroids.

This model has been used for practical applications by some researchers in the last years. For example, [19] investigated the stability of the problem and tested the results in a real Sun-Jupiter- (624) Hektor-spacecraft system; [20] studied periodic solutions of the problem in the Sun-Jupiter-Trojan Asteroid-spacecraft system.

This paper has attempted to analyze the motion of an infinitesimal mass in the gravitational field of the three primaries in the presence of small perturbations. Here, the dominant primary is modeled as a triaxial rigid body and the smaller one as an oblate spheroid. As to our knowledge there is not any similar work by other researchers in the international bibliography concerning the effects of the shapes (oblateness and triaxiality) of the primaries as applied to this problem.

The structure of the paper is as follows: Section 2 presents the equations of motion and the Jacobi integral of the system. Section 3 determines the equilibrium points, while Section 4 investigates their linear stability. Section 5 discusses the obtained results and conclusion of the paper.

2 Equations of Motion

Let m_1, m_2 and m_3 be the masses of three bodies, called hereafter the primaries, with $m_1 >> m_2 = m_3$ moving in circular periodic orbits around their center of mass fixed at the origin of the coordinates. These masses always lie at the vertices of equilateral triangle with the dominant body m_1 being on the negative *x*-axis at the origin of time. A massless particle is moving under the Newtonian gravitational attraction of the primaries and does not affect the motion of the three bodies. The motion of the system is referred to axes rotating with uniform angular velocity. The mutual distances of the three primaries remain unchanged with respect to time. This system is also dimensionless, i.e., the units of measure of length, mass and time are taken so that the sum of the masses, the distance between the primaries and the angular velocity is unity, which sets the gravitational constant G = 1. The factors characterizing the triaxiality of the dominant primary (m_1) and the oblateness coefficient of the smaller primary (m_2) are also taken into account.

Let the coordinates of the infinitesimal mass be (x, y) and masses m_1 , m_2 and m_3 are given by the relations:

$$(x_1, y_1) = \left(-\sqrt{3}\mu, 0\right),\ (x_2, y_2) = \left(\frac{\sqrt{3}}{2}(1-2\mu), -\frac{1}{2}\right),\ (x_3, y_3) = (x_2, -y_2).$$

We present the equations of motion of the problem following [8,12] in the usual dimensionless rectangular rotating coordinate system as:

$$\ddot{x} - 2n\dot{y} = \Omega_x, \tag{1}$$

$$\ddot{y} + 2n\dot{x} = \Omega_y, \tag{2}$$

where,

$$\Omega = \frac{n^2(x^2+y^2)}{2} + \frac{(1-2\mu)}{r_1} + \frac{\mu}{r_2} + \frac{\mu}{r_3} + \frac{(1-2\mu)(2\sigma_1-\sigma_2)}{2r_1^3} - \frac{3(1-2\mu)(\sigma_1-\sigma_2)y^2}{2r_1^5} + \frac{\mu A_2}{2r_2^3}\ , \tag{3}$$

with

$$r_1 = \sqrt{(x+\sqrt{3}\mu)^2 + \mathrm{y}^2},$$

$$r_2 = \sqrt{(x-\frac{\sqrt{3}}{2}(1-2\mu))^2 + (\mathrm{y}+\frac{1}{2})^2},$$

$$r_3 = \sqrt{(x-\frac{\sqrt{3}}{2}(1-2\mu))^2 + (\mathrm{y}-\frac{1}{2})^2},$$

and the mean motion n is given by

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}A_2 \tag{4}$$

Here Ω is the force gravitational potential, dots denote time derivatives, $r_i\,(i=1,2,3)$ are the distances between the fourth particle and the primaries, A_2 and $\sigma_i\,(i=1,2)$ are the oblateness and triaxiality factors respectively. Then $0 < A_2 << 1$ and $0 < \sigma_{1,2} << 1$. Therefore, we take $0 < \mu = \dfrac{m_2}{m_1+m_2+m_3} = \dfrac{m_3}{m_1+m_2+m_3} < \dfrac{1}{2}$, where μ is called the mass parameter of the problem.

The equations of motion admits the first integral (C is the Jacobian constant),

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C \tag{5}$$

2.1 Linear stability of the Lagrange configuration

The linear stability of the Lagrange central configuration is given by the inequality,

$$\frac{m_1m_2 + m_2m_3 + m_1m_3}{(m_1+m_2+m_3)^2} < \frac{1}{27}, \tag{6}$$

which was first studied by [21] in his thesis and later by [22] in the case of homogeneous potentials. Then using the idea of [14,23], if someone replaces the masses $m_i = (1-\sigma_i)(1-A_i)m_i$, $i=1,2,3$, then we believe that it will be produced the necessary condition for the stability of the Lagrange central configuration, in the present problem, i.e.,

$$\frac{am_1bm_2 + bm_2cm_3 + am_1cm_3}{\left(am_1+bm_2+cm_3\right)^2} < \frac{1}{27} \tag{7}$$

where we have abbreviated

$$a = (1-\sigma_1)(1-A_1), \qquad b = (1-\sigma_2)(1-A_2) \text{ and } \qquad c = (1-\sigma_3)(1-A_3)$$

Now, if we take $\sigma_1 = \sigma_2 = \sigma_3 = A_1 = A_2 = A_3 = 0$, we obtain inequality (6) of the linear stability of the Lagrange configuration [21].

In our present case where the dominant primary body m_1 is a triaxial rigid body and the second smaller primary body m_2 is an oblate spheroid, the problem admits inequality of the form

$$\frac{(1-\sigma_1)bm_1m_2 + bm_2m_3 + (1-\sigma_1)m_1m_3}{\left((1-\sigma_1)m_1 + bm_2 + m_3\right)^2} < \frac{1}{27} \tag{8}$$

In present work, we will consider sets of ($m_i, A_2, \sigma_1, \sigma_2$) which satisfy the condition (8). So, we assume that we have a dominant primary body with mass $m_1 = 0.97$ and two small equal primaries with masses $m_2 = m_3 = \mu = 0.015$.

3 Location and Existence of Equilibrium Points

The equilibrium points represent stationary solutions of the R4BP. These solutions are the singularities of the manifold of the components of the velocity and the coordinates and are found by setting $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0$ in the equations of motion. That is, they are the solutions of the equations $\Omega_x = \Omega_y = 0$, which are

$$n^2 x - \frac{(1-2\mu)(x+\sqrt{3}\mu)}{r_1^3} - \frac{\mu(x - \frac{\sqrt{3}}{2}(1-2\mu))}{r_2^3} - \frac{\mu(x - \frac{\sqrt{3}}{2}(1-2\mu))}{r_3^3} - \frac{3(2\sigma_1 - \sigma_2)(1-2\mu)(x+\sqrt{3}\mu)}{2r_1^5} -$$
$$\frac{3A_2\mu(x - \frac{\sqrt{3}}{2}(1-2\mu))}{2r_2^5} + \frac{15(\sigma_1 - \sigma_2)(1-2\mu)(x+\sqrt{3}\mu)y^2}{2r_1^7} = 0 \tag{9}$$

and

$$n^2 y - \frac{(1-2\mu)y}{r_1^3} - \frac{\mu(y+\frac{1}{2})}{r_2^3} - \frac{\mu(y-\frac{1}{2})}{r_3^3} - \frac{3(2\sigma_1 - \sigma_2)(1-2\mu)y}{2r_1^5} - \frac{3(1-2\mu)(\sigma_1 - \sigma_2)}{2}\left[\frac{2y}{r_1^5} - \frac{5y^3}{r_1^7}\right] -$$
$$\frac{3A_2\mu(y+\frac{1}{2})}{2r_2^5} = 0 \tag{10}$$

The solutions are categorized as follows:

3.1 Collinear equilibria

The collinear points are the solutions of (9) and (10) for x when $y = 0$. If $y = 0$, (10) is not fulfilled (since $A_2 \neq 0$). Thus, the solutions of (9) will not correspond to equilibrium points on the x-axis, called collinear equilibrium points. In the present problem, collinear equilibrium points do not exist under the combine effects of triaxiality and oblateness of the dominant and small primaries, respectively. We mention here that in the case where $A_2 = 0$, these points do exist and this may be directly seen from (10). So in this problem there are cases where collinear equilibria do not exist.

3.2 Non-collinear equilibrium points

The solutions of (9) and (10) for x and y with $y \neq 0$ give the location of the non-collinear points of the present problem. That is,

$$f(x,y) = \quad n^2 x - \frac{(1-2\mu)(x+\sqrt{3}\mu)}{r_1^3} - \frac{\mu(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_2^3} - \frac{\mu(x-\frac{\sqrt{3}}{2}(1-2\mu))}{r_3^3} -$$

$$\frac{3(2\sigma_1-\sigma_2)(1-2\mu)(x+\sqrt{3}\mu)}{2r_1^5} - \frac{3A_2\mu(x-\frac{\sqrt{3}}{2}(1-2\mu))}{2r_2^5} + \frac{15(\sigma_1-\sigma_2)(1-2\mu)(x+\sqrt{3}\mu)y^2}{2r_1^7} = 0 \tag{11}$$

$$g(x,y) = n^2 y - \frac{(1-2\mu)y}{r_1^3} - \frac{\mu(y+\frac{1}{2})}{r_2^3} - \frac{\mu(y-\frac{1}{2})}{r_3^3} - \frac{3(2\sigma_1-\sigma_2)(1-2\mu)y}{2r_1^5} -$$

$$\frac{3(1-2\mu)(\sigma_1-\sigma_2)}{2}\left[\frac{2y}{r_1^5} - \frac{5y^3}{r_1^7}\right] - \frac{3A_2\mu(y+\frac{1}{2})}{2r_2^5} = 0 \tag{12}$$

In Fig. 1, we illustrate the eight non-collinear equilibrium points L_i, i =1,…,8 of the problem for $\mu = 0.015$, $\sigma_1 = 2.284\times10^{-12}, \sigma_2 = 1.141\times10^{-12}$ and A_2 = 0.01, which are found by solving numerically (11) and (12). The red curve represents the $f(x,y)$ and blue curve the $g(x,y)$. We opted to name them in the same way as in the work of [14]. One can easily see eight points of intersection of the curves (blue and red), which corresponds to eight equilibrium positions of the infinitesimal body m. The positions of the primary bodies m_i are denoted by large black points while the positions of the equilibrium points L_i are denoted by small black dots.

In order to investigate the effect of the triaxiality of the dominant primary body m_1 and oblateness of the smaller body m_2 on the positions of the equilibrium points, we give the numerical positions of the points for varying triaxiality of the dominant primary and oblateness of the smaller one. In Tables 1—3, the positions of the equilibrium points of the problem are presented for fixed values of triaxiality and increasing values of oblateness parameter. Figs. 2.1 and 2.2 show the evolution of the equilibrium points in (x, A_2) and (y, A_2) plane, respectively, as A_2 varies for fixed values of triaxiality factor $\sigma_i (i=1,2)$. The numbers 1—3 correspond to the value (σ_1, σ_2) = $(2.284\times10^{-12}, 1.141\times10^{-12})$, (0.025, 0.015), (0.085, 0.065), respectively. An investigation of this shows that as the oblateness factor varies for fixed values of triaxial factor, the positions of the equilibrium points are significantly affected.

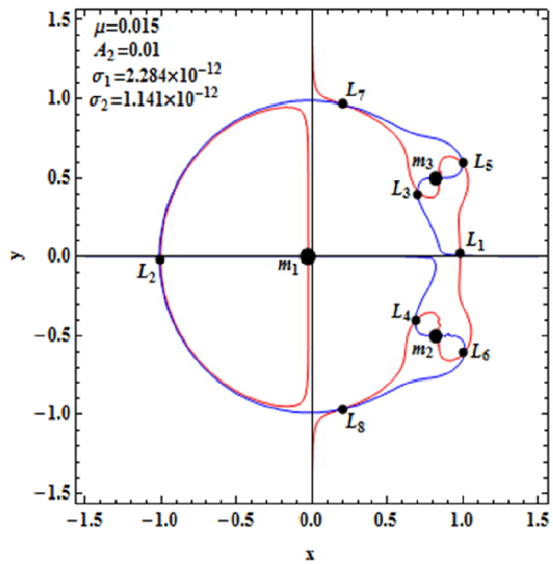

Fig. 1. The positions of the equilibrium points L_i, i =1,…,8 of the problem for $\mu = 0.015$, $\sigma_1 = 2.284\times10^{-12}, \sigma_2 = 1.141\times10^{-12}$ and A_2= 0.01. The red curve represents the $f(x, y)$ and blue curve the $g(x, y)$. The mass distribution is $m_1 = 0.97$ and $m_2 = m_3 = \mu = 0.015$. With large black points and small dots, we indicate the primary bodies and equilibrium points of the problem correspondingly

4 Linear Stability of Non-collinear Equilibrium Points

In order to study their stability we transfer the origin at an equilibrium points $L_i, i = 1,\cdots,8$:

$$x = x_0 + \xi,$$
$$\mathrm{y} = \mathrm{y}_0 + \eta, \tag{13}$$

and we expand the equations of motion (1) and (2) into first-order terms with respect to ξ and η obtaining the linearized system:

$$\dot{\bar{x}} = \boldsymbol{A}\,\bar{x}\ ,\quad \bar{x} = (\xi,\eta,\dot{\xi},\dot{\eta})^T \tag{14}$$

where $\bar{x}$ is the state vector of the fourth particle with respect to the equilibrium points and the bold typeface ***A*** denotes the time-independent coefficient matrix and has the form

$$\boldsymbol{A}= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega^0_{xx} & \Omega^0_{xy} & 0 & 2n \\ \Omega^0_{yx} & \Omega^0_{yy} & -2n & 0 \end{pmatrix}, \tag{15}$$

with

$$\Omega_{xx}^{0}=n^{2}-\frac{(1-2\mu)}{r_{10}^{3}}-\frac{\mu}{r_{20}^{3}}-\frac{\mu}{r_{30}^{3}}+\frac{3(1-2\mu)}{2r_{10}^{5}}\left[2(x_{0}+\sqrt{3}\mu)^{2}-(2\sigma_{1}-\sigma_{2})\right]+\frac{3\mu}{2r_{20}^{5}}\left[2(x_{0}-\frac{\sqrt{3}}{2}(1-2\mu))^{2}-A_{2}\right]+$$
$$\frac{3\mu(x_{0}-\frac{\sqrt{3}}{2}(1-2\mu))^{2}}{r_{30}^{5}}+\frac{15(2\sigma_{1}-\sigma_{2})(1-2\mu)(x_{0}+\sqrt{3}\mu)^{2}}{2r_{10}^{7}}+\frac{15(\sigma_{1}-\sigma_{2})(1-2\mu){y_{0}}^{2}}{2}\left[\frac{1}{r_{10}^{7}}-\frac{7(x_{0}+\sqrt{3}\mu)^{2}}{r_{10}^{9}}\right]+$$
$$\frac{15A_{2}\mu(x_{0}-\frac{\sqrt{3}}{2}(1-2\mu)^{2}}{2r_{20}^{7}} \tag{16}$$

$$\Omega_{yy}^{0}=n^{2}-\frac{(1-2\mu)}{r_{10}^{3}}-\frac{\mu}{r_{20}^{3}}-\frac{\mu}{r_{30}^{3}}-\frac{3(1-2\mu)}{2r_{10}^{5}}\left[(2\sigma_{1}-\sigma_{2})-2{y_{0}}^{2}\right]-\frac{3(\sigma_{1}-\sigma_{2})(1-2\mu)}{r_{10}^{5}}-\frac{3\mu}{2r_{20}^{5}}\left[A_{2}-2(y_{0}+\frac{1}{2})^{2}\right]+$$
$$\frac{3\mu(y_{0}-\frac{1}{2})^{2}}{r_{30}^{5}}+\frac{15(1-2\mu){y_{0}}^{2}}{2r_{10}^{7}}(7\sigma_{1}-6\sigma_{2})+\frac{15A_{2}\mu(y_{0}+\frac{1}{2})^{2}}{2r_{20}^{7}}-\frac{105(\sigma_{1}-\sigma_{2})(1-2\mu){y_{0}}^{4}}{2r_{10}^{9}} \tag{17}$$

$$\Omega_{xy}^{0}=\Omega_{yx}^{0}=\frac{3(1-2\mu)(x_{0}+\sqrt{3}\mu)y_{0}}{r_{10}^{5}}+3\mu(x_{0}-\frac{\sqrt{3}}{2}(1-2\mu))\left[\frac{(y_{0}+\frac{1}{2})}{r_{20}^{5}}+\frac{(y_{0}-\frac{1}{2})}{r_{30}^{5}}\right]+$$
$$\frac{15(1-2\mu)(x_{0}+\sqrt{3}\mu)y_{0}}{2r_{10}^{7}}(4\sigma_{1}-3\sigma_{2})+\frac{15A_{2}\mu(y_{0}+\frac{1}{2})(x_{0}-\frac{\sqrt{3}}{2}(1-2\mu))}{2r_{20}^{7}}-\frac{105(\sigma_{1}-\sigma_{2})(1-2\mu)(x_{0}+\sqrt{3}\mu){y_{0}}^{3}}{2r_{10}^{9}} \tag{18}$$

where the superscript ‘ 0 ‘ indicates that these derivatives are evaluated at the equilibrium point (x_0, y_0).

with

$$r_{10}=\sqrt{(x_{0}+\sqrt{3}\mu)^{2}+{\mathrm{y}_{0}}^{2}},$$
$$r_{20}=\sqrt{(x_{0}-\frac{\sqrt{3}}{2}(1-2\mu))^{2}+(\mathrm{y}_{0}+\frac{1}{2})^{2}},$$
$$r_{30}=\sqrt{(x_{0}-\frac{\sqrt{3}}{2}(1-2\mu))^{2}+(\mathrm{y}_{0}-\frac{1}{2})^{2}},$$

The characteristic equation of the matrix ***A*** is given by

$$\lambda^{4}+a\lambda^{2}+b=0 \tag{19}$$

with:

$$a=4n^{2}-\Omega_{xx}^{0}-\Omega_{yy}^{0}$$
$$b=\Omega_{xx}^{0}\Omega_{yy}^{0}-(\Omega_{xy}^{0})^{2}$$

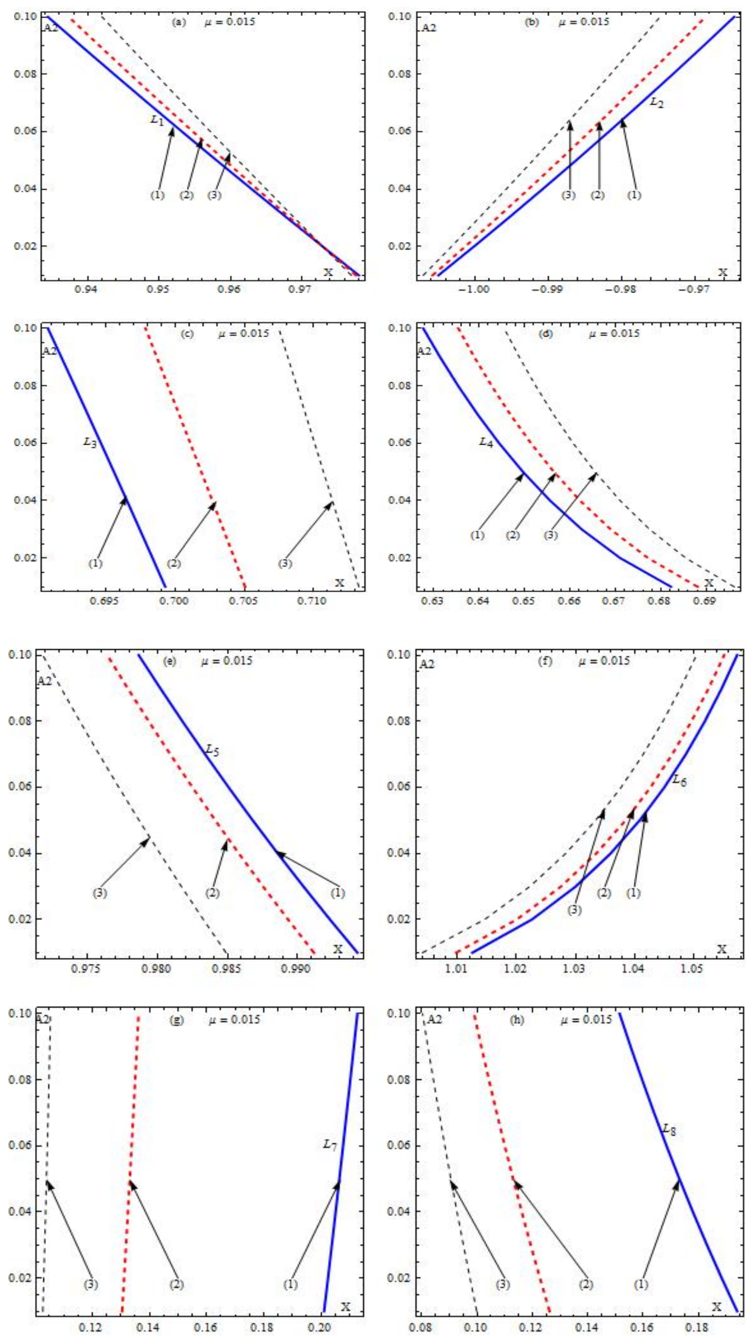

Fig. 2.1. Frames (a) to (h). The positions of the equilibrium points $L_i, i = 1,\ldots,8$ in the (x, A_2) plane as the oblateness factor varies for fixed values of triaxial factor: (1) $\sigma_1 = 2.284\times10^{-12}, \sigma_2 = 1.141\times10^{-12}$, (2) $\sigma_1 = 0.025, \sigma_2 = 0.015$ and (3) $\sigma_1 = 0.085, \sigma_2 = 0.065$. The mass distribution is $m_1 = 0.97$ and $m_2 = m_3 = \mu = 0.015$

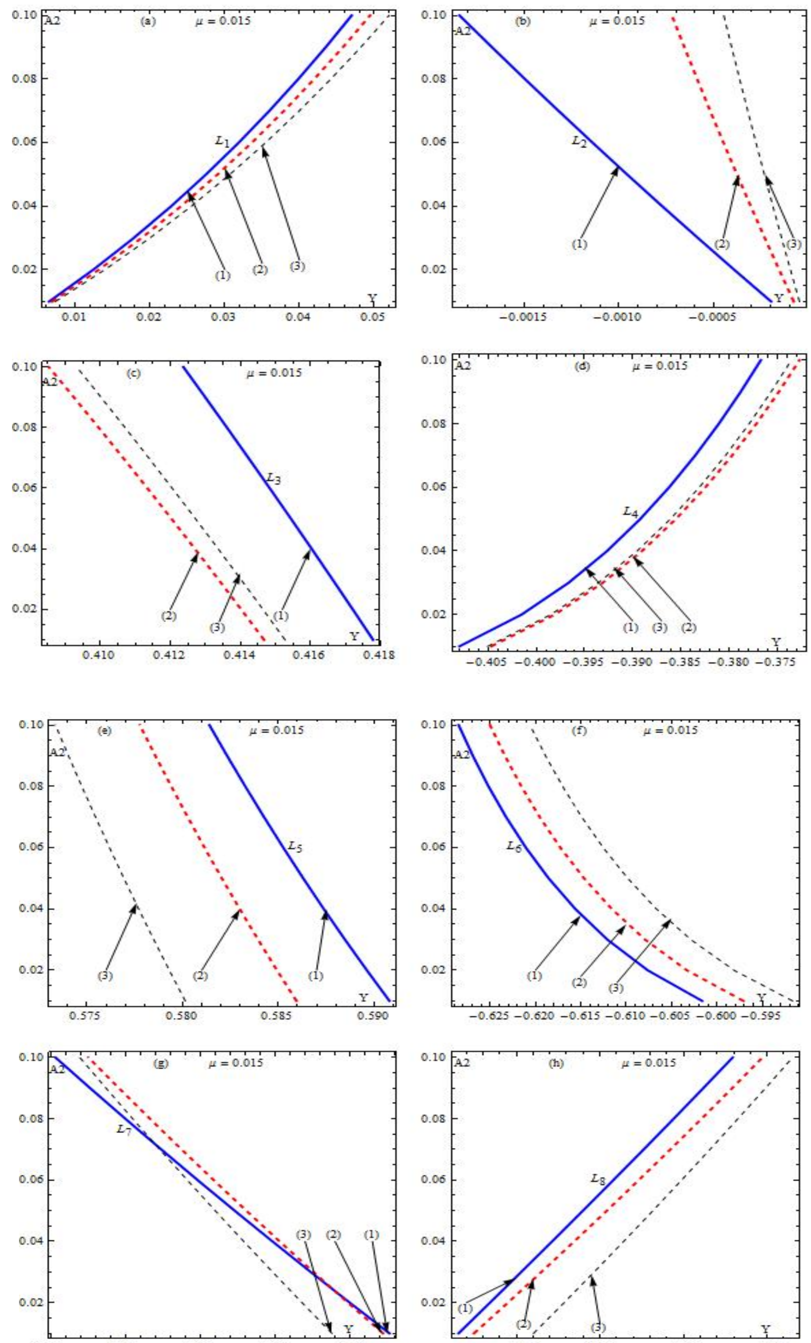

Fig. 2.2. Frames (a) to (h). The positions of the equilibrium points $L_i, i = 1,...,8$ in the (y, A_2) plane as the oblateness factor varies for fixed values of triaxial factor: (1) $\sigma_1 = 2.284\times10^{-12}, \sigma_2 = 1.141\times10^{-12}$, (2) $\sigma_1 = 0.025, \sigma_2 = 0.015$ and (3) $\sigma_1 = 0.085, \sigma_2 = 0.065$. The mass distribution is $m_1 = 0.97$ and $m_2 = m_3 = \mu = 0.015$

Table 1. Numerical computations of non-collinear points for μ = 0.015, $\sigma_1 = 2.284\times10^{-12}, \sigma_2 = 1.141\times10^{-12}$ and 0 < A_2 <<1

σ_1=**2.284**$\times10^{-12}$
σ_2=**1.141**$\times10^{-12}$

A_2	L_1	L_2	L_3	L_4
0.01	(0.978020, 0.0064824)	(-1.00496, -0.000198)	(0.699266, 0.417790)	(0.682226,-0.407977)
0.02	(0.972963, 0.0123870)	(-1.00014,- 0.000393)	(0.698351, 0.417212)	(0.671051,-0.401531)
0.03	(0.967950, 0.0178113)	(-0.99541, -0.000585)	(0.697428, 0.416628)	(0.662572,-0.396652)
0.04	(0.962986, 0.0228293)	(-0.99076, -0.000773)	(0.696499, 0.416038)	(0.655613,-0.392656)
0.05	(0.958074, 0.0274987)	(-0.98621, -0.000958)	(0.695563, 0.415442)	(0.649646,-0.389236)
0.06	(0.953219, 0.0318658)	(-0.98174, -0.001141)	(0.694621, 0.414840)	(0.644386,-0.386227)
0.07	(0.948422, 0.0359683)	(-0.97735, -0.001320)	(0.693672, 0.414232)	(0.639659,-0.383527)
0.08	(0.943683, 0.0398372)	(-0.97304, -0.001497)	(0.692717, 0.413618)	(0.635349,-0.381070)
0.09	(0.939004, 0.0434985)	(-0.96880, -0.001671)	(0.691756, 0.412998)	(0.631377,-0.378810)
0.10	(0.934384, 0.0469745)	(-0.96464, -0.001842)	(0.690789, 0.412372)	(0.627685, -0.376711)

Table 1 continued

L_5	L_6	L_7	L_8
(0.994278, 0.590801)	(1.01251, -0.601645)	(0.201145, 0.965276)	(0.193931,-0.966915)
(0.992323, 0.589620)	(1.02259, -0.607642)	(0.202502, 0.959968)	(0.188364,-0.963163)
(0.990429, 0.588477)	(1.03003, -0.612078)	(0.203843, 0.954752)	(0.183048,-0.959431)
(0.988593, 0.587369)	(1.03595, -0.615613)	(0.205168, 0.949627)	(0.177963,-0.955724)
(0.986813, 0.586296)	(1.04087, -0.618550)	(0.206476, 0.944589)	(0.173092,-0.952043)
(0.985087, 0.585256)	(1.04506, -0.621060)	(0.207769, 0.939636)	(0.168419,-0.948392)
(0.983411, 0.584248)	(1.04871, -0.623247)	(0.209047, 0.934764)	(0.163932,-0.944773)
(0.981784, 0.583269)	(1.05194, -0.625182)	(0.210310, 0.929972)	(0.159617,-0.941186)
(0.980204, 0.582319)	(1.05482, -0.626913)	(0.211559, 0.925256)	(0.155463,-0.937634)
(0.978668, 0.581396)	(1.05742, -0.628475)	(0.212794, 0.920615)	(0.151462, -0.934118)

Table 2. Numerical computations of non-collinear points for μ = 0.015, $\sigma_1 = 0.025, \sigma_2 = 0.015$ and 0 < A_2 <<1

σ_1=0.025 σ_2=0.015				
A_2	L_1	L_2	L_3	L_4
0.01	(0.977607, 0.0068880)	(-1.00576, -0.000079)	(0.705040, 0.414677)	(0.688095,-0.404665)
0.02	(0.972963, 0.0131368)	(-1.00132,- 0.000156)	(0.704256, 0.414027)	(0.677183,-0.398085)
0.03	(0.968350, 0.0188596)	(-0.99696, -0.000232)	(0.703467, 0.413369)	(0.668948,-0.393108)
0.04	(0.963774, 0.0241411)	(-0.99269, -0.000306)	(0.702674, 0.412703)	(0.662214,-0.389029)
0.05	(0.959242, 0.0290467)	(-0.98849, -0.000378)	(0.701877, 0.412029)	(0.656457,-0.385535)
0.06	(0.954755, 0.0336282)	(-0.98436, -0.000450)	(0.701077, 0.411346)	(0.651394,-0.382457)
0.07	(0.950317, 0.0379272)	(-0.98031, -0.000520)	(0.700273, 0.410654)	(0.646854,-0.379693)
0.08	(0.945929, 0.0419779)	(-0.97632, -0.000588)	(0.699466, 0.409955)	(0.642724,-0.377175)
0.09	(0.941591, 0.0458089)	(-0.97241, -0.000656)	(0.698657, 0.409246)	(0.638925,-0.374855)
0.10	(0.937305, 0.0494441)	(-0.96856, -0.000722)	(0.697844, 0.408529)	(0.635399, -0.372699)

Table 2 continued

L_5	L_6	L_7	L_8
(0.991173, 0.585944)	(1.00976, -0.597093)	(0.130471, 0.964410)	(0.126167,-0.965073)
(0.989334, 0.584926)	(1.02000, -0.603323)	(0.131161, 0.959719)	(0.122698,-0.961015)
(0.987550, 0.583938)	(1.02753, -0.607926)	(0.131838, 0.955112)	(0.119352,-0.957015)
(0.985819, 0.582979)	(1.03351, -0.611595)	(0.132502, 0.950588)	(0.116123,-0.953072)
(0.984139, 0.582047)	(1.03848, -0.614649)	(0.133153, 0.946142)	(0.113005,-0.949185)
(0.982507, 0.581142)	(1.04272, -0.617262)	(0.133791, 0.941773)	(0.109991,-0.945353)
(0.980922, 0.580261)	(1.04641, -0.619544)	(0.134418, 0.937479)	(0.107076,-0.941575)
(0.979382, 0.579405)	(1.04967, -0.621565)	(0.135032, 0.933258)	(0.104256,-0.937850)
(0.977884, 0.578572)	(1.05259, -0.623376)	(0.135635, 0.929106)	(0.101526,-0.934177)
(0.976426, 0.577761)	(1.05522, -0.625015)	(0.136226, 0.925023)	(0.098881, -0.930556)

Table 3. Numerical computations of non-collinear points for μ = 0.015, $\sigma_1 = 0.085, \sigma_2 = 0.065$ and 0 < A_2 <<1

σ_1 =0.085 σ_2 =0.065				
A_2	L_1	L_2	L_3	L_4
0.01	(0.976961, 0.0073252)	(-1.00702, -0.000048)	(0.713264, 0.415278)	(0.696153,-0.405055)
0.02	(0.972969, 0.0139393)	(-1.00319,- 0.000096)	(0.712627, 0.414649)	(0.685487,-0.398495)
0.03	(0.968991, 0.0199748)	(-0.99942, -0.000143)	(0.711989, 0.414012)	(0.677514,-0.393568)
0.04	(0.965034, 0.0255291)	(-0.99572, -0.000189)	(0.711349, 0.413368)	(0.671031,-0.389547)
0.05	(0.961104, 0.0306764)	(-0.99208, -0.000234)	(0.710707, 0.412715)	(0.665513,-0.366112)
0.06	(0.957207, 0.0354745)	(-0.98850, -0.000278)	(0.710063, 0.412054)	(0.660677,-0.383093)
0.07	(0.953344, 0.0399697)	(-0.98498, -0.000322)	(0.709417, 0.411385)	(0.656353,-0.380385)
0.08	(0.949518, 0.0441997)	(-0.98151, -0.000364)	(0.708771, 0.410708)	(0.652430,-0.377922)
0.09	(0.945730, 0.0481956)	(-0.97810, -0.000406)	(0.708122, 0.410022)	(0.648828,-0.375655)
0.10	(0.941982,0.0519833)	(-0.97474, -0.000448)	(0.707473, 0.409328)	(0.645493, -0.373551)

Table 3 continued

L_5	L_6	L_7	L_8
(0.984905, 0.580134)	(1.00405, -0.591669)	(0.102717, 0.957580)	(0.099857,-0.957944)
(0.983289, 0.579302)	(1.01455, -0.598130)	(0.103052, 0.953583)	(0.097417,-0.954296)
(0.981718, 0.578491)	(1.02224, -0.602887)	(0.103377, 0.949653)	(0.095050,-0.950702)
(0.980189, 0.577701)	(1.02834, -0.606677)	(0.103695, 0.945788)	(0.092753,-0.947161)
(0.978701, 0.576930)	(1.03340, -0.609836)	(0.104004, 0.941987)	(0.090522,-0.943670)
(0.977253, 0.576179)	(1.03773, -0.612543)	(0.104305, 0.938248)	(0.088356,-0.940230)
(0.975843, 0.575447)	(1.04149, -0.614911)	(0.104598, 0.934568)	(0.086251,-0.936839)
(0.974470, 0.574732)	(1.04483, -0.617013)	(0.104884, 0.930948)	(0.084204,-0.933496)
(0.973131, 0.574034)	(1.04782, -0.618902)	(0.105162, 0.927384)	(0.082214,-0.930199)
(0.971827, 0.573354)	(1.05052, -0.620613)	(0.105433, 0.923875)	(0.080278, -0.926948)

The stability of the non-collinear equilibrium points under the joint effects of triaxiality of the first primary and oblateness of the second primary are determined by the roots of the characteristic equation (19). Stability occurs when all roots of the characteristic equation for λ are purely imaginary. Therefore, for stability the following three conditions must be fulfilled simultaneously:

$$(4n^2 - \Omega^o_{xx} - \Omega^o_{yy}) > 0,$$
$$(\Omega^o_{xx}\Omega^o_{yy} - (\Omega^o_{xy})^2) > 0, \quad (20)$$
$$(4n^2 - \Omega^o_{xx} - \Omega^o_{yy})^2 - 4\,(\Omega^o_{xx}\Omega^o_{yy} - (\Omega^o_{xy})^2) > 0$$

The eigenvalues of (19) determine the stability or instability of the equilibrium points. Studying the stability of the non-collinear equilibrium points of the problem as the oblateness and triaxial factors varies for the numerical examples given in Tables 1, 2 and 3, we found that point L_1 is unstable as seen in Tables 4, 5 and 6 since for L_1, the characteristic equation has four eigenvalues of the form $\lambda_{1,2,3,4} = \pm a \pm ib$ while points L_2, L_3, L_4, L_5, L_6 are unstable since for $L_i, i = 2,\cdots,6$, the characteristic equation has two real eigenvalues $\lambda_{1,2} = \pm a$ and two imaginary eigenvalues $\lambda_{3,4} = \pm ib$. However, points L_7 and L_8 are affected from the involved parameters as one found that for values near the gravitational case (Table 4), the equilibrium points are stable, since it has pure imaginary roots of the form $\lambda_{1,2} = \pm ai$, $\lambda_{3,4} = \pm ib$; while unstable for the specific numerical example given in Tables 5 and 6 since it has four eigenvalues of the form $\lambda_{1,2,3,4} = \pm a \pm ib$.

Table 4. The characteristic roots of the non-collinear points $L_i, i = 1,\ldots,8$ as a function of oblateness parameter A_2 for $\mu = 0.015, \sigma_1 = 2.284\times10^{-12}, \sigma_2 = 1.141\times10^{-12}$

A_2	$\lambda_{1,2}[L_1]$	$\lambda_{3,4}[L_1]$	$\lambda_{1,2}\ [L_2]\,\lambda_{3,4}$
0.01	$0.595382 \pm 0.879906\,i$	$-0.595382 \pm 0.879906\,i$	± 0.255090, $\pm 1.028063\,i$
0.04	$0.642216 \pm 0.911113\,i$	$-0.642216 \pm 0.911113\,i$	± 0.262958, $\pm 1.050947\,i$
0.07	$0.680983 \pm 0.939182\,i$	$-0.680983 \pm 0.939182\,i$	± 0.270578, $\pm 1.073338\,i$
0.10	$0.714443 \pm 0.964994\,i$	$-0.714443 \pm 0.964994\,i$	± 0.278278, $\pm 1.095317\,i$
A_2	$\lambda_{1,2}\ [L_3]\,\lambda_{3,4}$	$\lambda_{1,2}\ [L_4]\,\lambda_{3,4}$	$\lambda_{1,2}\ [L_5]\,\lambda_{3,4}$
0.01	± 2.914077, $\pm 2.320718\,i$	± 3.313719, $\pm 2.331891\,i$	± 2.160432, $\pm 1.864816\,i$
0.04	± 2.837039, $\pm 2.282654\,i$	± 3.618663, $\pm 2.357304\,i$	± 2.289597, $\pm 1.953605\,i$
0.07	± 2.761955, $\pm 2.246380\,i$	± 3.724328, $\pm 2.377436\,i$	± 2.418734, $\pm 2.042584\,i$
0.10	± 2.688882, $\pm 2.211915\,i$	± 3.779053, $\pm 2.393337\,i$	± 2.547760, $\pm 2.131677\,i$
A_2	$\lambda_{1,2}\ [L_6]\,\lambda_{3,4}$	$\lambda_{1,2}\ [L_7]\,\lambda_{3,4}$	$\lambda_{1,2}\ [L_8]\,\lambda_{3,4}$
0.01	± 2.389957, $\pm 1.789116\,i$	$\pm 0.850721\,i$, $\pm 0.526694\,i$	$\pm 0.846737\,i$, $\pm 0.531837\,i$
0.04	± 2.704491, $\pm 1.786679\,i$	$\pm 0.862591\,i$, $\pm 0.548655\,i$	$\pm 0.846373\,i$, $\pm 0.568939\,i$
0.07	± 2.883090, $\pm 1.816587\,i$	$\pm 0.873704\,i$, $\pm 0.570656\,i$	$\pm 0.844167\,i$, $\pm 0.606538\,i$
0.10	± 3.024419, $\pm 1.853565\,i$	$\pm 0.884058\,i$, $\pm 0.592739\,i$	$\pm 0.838925\,i$, $\pm 0.646028\,i$

Table 5. The characteristic roots of the non-collinear points $L_i, i = 1,\ldots,8$ as a function of oblateness parameter A_2 for $\mu = 0.015, \sigma_1 = 0.025, \sigma_2 = 0.015$

A_2	$\lambda_{1,2}[L_1]$	$\lambda_{3,4}[L_1]$	$\lambda_{1,2}\ [L_2]\ \lambda_{3,4}$
0.01	$0.605828 \pm 0.881990i$	$-0.605828 \pm 0.881990i$	$\pm 0.407940, \quad \pm 1.062459i$
0.04	$0.653451 \pm 0.913142i$	$-0.653451 \pm 0.913142i$	$\pm 0.421008, \quad \pm 1.085160i$
0.07	$0.692714 \pm 0.940941i$	$-0.692714 \pm 0.940941i$	$\pm 0.433975, \quad \pm 1.107432i$
0.10	$0.726624 \pm 0.966395i$	$-0.726624 \pm 0.966395i$	$\pm 0.446830, \quad \pm 1.129297i$
A_2	$\lambda_{1,2}\ [L_3]\ \lambda_{3,4}$	$\lambda_{1,2}\ [L_4]\ \lambda_{3,4}$	$\lambda_{1,2}\ [L_5]\ \lambda_{3,4}$
0.01	$\pm 3.029502, \pm 2.392471i$	$\pm 3.458523, \pm 2.409353i$	$\pm 2.266850, \pm 1.939158i$
0.04	$\pm 2.954834, \pm 2.354766i$	$\pm 3.791779, \pm 2.444376i$	$\pm 2.394479, \pm 2.027103i$
0.07	$\pm 2.882023, \pm 2.318691i$	$\pm 3.912530, \pm 2.470314i$	$\pm 2.522064, \pm 2.115203i$
0.10	$\pm 2.811156, \pm 2.284280i$	$\pm 3.978634, \pm 2.490611i$	$\pm 2.649498, \pm 2.203371i$
A_2	$\lambda_{1,2}\ [L_6]\ \lambda_{3,4}$	$\lambda_{1,2}\ [L_7]$	$\lambda_{3,4}\ [L_7]$
0.01	$\pm 2.511320, \pm 1.857737i$	$0.089891 \pm 0.526694i$	$-0.089891 \pm 0.715421i$
0.04	$\pm 2.825062, \pm 1.849965i$	$0.104136 \pm 0.731460i$	$-0.104136 \pm 0.731460i$
0.07	$\pm 2.999071, \pm 1.876805i$	$0.116817 \pm 0.747146i$	$-0.116817 \pm 0.747146i$
0.10	$\pm 3.136478, \pm 1.911573i$	$0.128402 \pm 0.762506i$	$-0.128402 \pm 0.762506i$
A_2	$\lambda_{1,2}\ [L_8]$	$\lambda_{3,4}\ [L_8]$	
0.01	$0.099283 \pm 0.716291i$	$-0.099283 \pm 0.716291i$	
0.04	$0.133676 \pm 0.734814i$	$-0.133676 \pm 0.734814i$	
0.07	$0.160801 \pm 0.752848i$	$-0.160801 \pm 0.752848i$	
0.10	$0.183963 \pm 0.770438i$	$-0.183963 \pm 0.770438i$	

Table 6. The characteristic roots of the non-collinear points $L_i, i = 1,\ldots,8$ as a function of oblateness parameter A_2 for $\mu = 0.015, \sigma_1 = 0.085, \sigma_2 = 0.065$

A_2	$\lambda_{1,2}[L_1]$	$\lambda_{3,4}[L_1]$	$\lambda_{1,2}\ [L_2]\ \lambda_{3,4}$
0.01	$0.641492 \pm 0.888810i$	$-0.641492 \pm 0.888810i$	$\pm 0.548567, \quad \pm 1.083498i$
0.04	$0.690965 \pm 0.920756i$	$-0.690965 \pm 0.913142i$	$\pm 0.564761, \quad \pm 1.104195i$
0.07	$0.729684 \pm 0.947327i$	$-0.729684 \pm 0.947327i$	$\pm 0.580687, \quad \pm 1.124489i$
0.10	$0.764261 \pm 0.972284i$	$-0.764261 \pm 0.972284i$	$\pm 0.596536, \quad \pm 1.144454i$
A_2	$\lambda_{1,2}\ [L_3]\ \lambda_{3,4}$	$\lambda_{1,2}\ [L_4]\ \lambda_{3,4}$	$\lambda_{1,2}\ [L_5]\ \lambda_{3,4}$
0.01	$\pm 3.286931, \pm 2.392471i$	$\pm 3.780243, \pm 2.409353i$	$\pm 2.459780, \pm 2.072539i$
0.04	$\pm 3.215585, \pm 2.354766i$	$\pm 4.165707, \pm 2.620513i$	$\pm 2.584186, \pm 2.158668i$
0.07	$\pm 3.145905, \pm 2.469826i$	$\pm 4.312432, \pm 2.655187i$	$\pm 2.708590, \pm 2.244935i$
0.10	$\pm 3.077951, \pm 2.434648i$	$\pm 4.397541, \pm 2.681879i$	$\pm 2.832847, \pm 2.331237i$
A_2	$\lambda_{1,2}\ [L_6]\ \lambda_{3,4}$	$\lambda_{1,2}\ [L_7]$	$\lambda_{3,4}\ [L_7]$
0.01	$\pm 2.733096, \pm 1.981541i$	$0.303005 \pm 0.748753i$	$-0.303005 \pm 0.748753i$
0.04	$\pm 3.046338, \pm 1.965046i$	$0.314708 \pm 0.763648i$	$-0.314708 \pm 0.763648i$
0.07	$\pm 3.213509, \pm 1.987150i$	$0.326186 \pm 0.778245i$	$-0.326186 \pm 0.778245i$

0.10	$\pm 3.343910, \pm 2.018032i$	$0.337460 \pm 0.792565i$	$-0.337460 \pm 0.792565i$
A_2	$\lambda_{1,2}$ [L_8]	$\lambda_{3,4}$ [L_8]	
0.01	$0.306023 \pm 0.749640i$	$-0.306023 \pm 0.749640i$	
0.04	$0.326196 \pm 0.767125i$	$-0.326196 \pm 0.767125i$	
0.07	$0.345394 \pm 0.784212i$	$-0.345394 \pm 0.784212i$	
0.10	$0.363779 \pm 0.800929i$	$-0.363779 \pm 0.800929i$	

5 Discussion and Conclusion

In this contribution, we have studied the restricted four-body problem when the primary bodies m_1, m_2 and m_3 are always at the vertices of an equilateral triangle (Lagrange equilateral triangle configuration) in the special case of two primaries with equal masses. The fourth particle in this system has negligible mass m with respect to the primaries, and its motion is perturbed by a triaxial rigid body ($\sigma_i, i = 1,2$) and oblateness coefficient A_2 from the primaries m_1 and m_2, respectively. We studied the existence, location and stability of the equilibrium points as the involved parameters varies. It has been found that, collinear equilibrium points do not exist and there are eight non-collinear points, named $L_i, i = 1,...,8$. In Tables 1—3 we present these positions as the parameters varies. These are shown graphically in Figs. 2.1 and 2.2. It is observed that there is a visible left (right) shift in the positions of the equilibrium points due to the involved parameters. Finally, the linear stability investigation has been achieved by determining the roots of the characteristic equation. It is noticed that the non-sphericity of the bodies does not affect the nature of the stability of the equilibrium points $L_i, i = 1,\ldots,6$ since they remain unstable (Tables 4—6) in the Lyapunov sense for the specific numerical examples given here, while the stability of the points L_7 and L_8 are seen to be affected for large deviations of its value from the gravitational case (Tables 5 and 6). It is remarkable to note that, equations of motion are unlike those obtained by [14,16] and references therein. Also, when these terms are neglected in the present problem, we get the same terms as in the classical restricted four-body problem (Lagrangian configuration).

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Lagrange JL. Essay on the problem of three bodies. Works of Lagrange. 1772;6:229-331.

[2] Laplace P, Delaunay B. Memoire sur la theorie de la Lune, Mem. Del'des Sci. 1867; 28-29. (French).

[3] Poincare H. Les Me'thodes Nouvelles de la Me'chanique Ce'leste. Gauthier-Villars, Paris; 1892. (French).

[4] Birkhoff GD. Dynamical systems. Amer. Math. Soc. Colloquium Publ. Rhode Island; 1927.

[5] Szebehely VG. Theory of orbits. The restricted problem of three bodies. Academic press, New York; 1976.

[6] Khanna M, Bhatnagar KB. Existence and stability of libration points in the restricted three-body problem when the smaller primary is a triaxial rigid body and the bigger one an oblate spheroid. International Journal of Pure and Applied Mathematics. 1999;30:721–733.

[7] Sharma RK, Taqvi ZA, Bhatnagar KB. Existence and stability of libration points in the restricted three-body problem when the primaries are triaxial rigid bodies. Celestial Mechanics and Dynamical Astronomy. 2001;79:119-126.

[8] Singh J, Begha JM. Stability of equilibrium points in the generalized perturbed restricted three-body problem. Astrophysics and Space Science. 2011;331:511-517.

[9] Singh J, Laraba HM. Robe’s circular restricted three-body problem under oblate and triaxial primaries. Earth Moon and Planets. 2012;109:1–11.

[10] Singh J, Umar A. The collinear libration points in the elliptic R3BP with a triaxial primary and an oblate secondary. International Journal of Astronomy and Astrophysics. 2014;4:391-398.

[11] Alvarez- Ramírez M, Vidal C. Dynamical aspects of an equilateral restricted four-body problem. Mathematical Problems in Engineering. 2009; 2009:1-23.

[12] Baltagiannis AN, Papadakis KE. Equilibrium points and their stability in the restricted four-body problem. Int. J. Birfurc. Chaos. 2011a;21:2179-2193.

[13] Burgos-Garcia J, Delgado J. Periodic orbits in the restricted four-body problem with two equal masses. Astrophys. Space Sci. 2013;345:247-263.

[14] Papadouris JP, Papadakis KE. Equilibrium points in the photogravitational restricted four-body problem. Astrophys. Space. Sci. 2013;344:21-38.

[15] Papadouris JP, Papadakis KE. Periodic solutions in the photogravitational restricted four-body problem. MNRAS. 2014;442:1628-1639.

[16] Kumari B, Kushvah BS. Stability regions of equilibrium points in restricted four-body problem with oblateness effects. Astrophys. Space Sci. 2014;349:693-704.

[17] Singh J, Vincent EA. Out-of-plane equilibrium points in the photogravitational restricted four-body problem. Astrophys. Space Sci., 2015. (In press) doi: 10.1007/s10509-015-2487-0

[18] Singh J, Vincent EA. Out-of-plane equilibrium points in the photogravitational restricted four-body problem with oblateness. British Journal of Mathematics and Computer Science. 2016;19(5):1-15.

[19] Ceccaroni M, Biggs J. Low-thrust propulsion in a coplanar circular restricted four-body problem. Celest. Mech. Dyn. Astron. 2012;112:191-219.

[20] Baltagiannis AN, Papadakis KE. Periodic solutions in the Sun-Jupiter-Trojan Asteroid-Spacecraft system. Planetary and Space Science. 2013;75:148-157.

[21] Gascheau M. Examen d’une classe d’equations differentielles et application a un cas particulier du probleme des trios corps. Comp. Rend. Acad. Sci. 1843;16:393-394. French.

[22] Routh EJ. On Laplace’s three particles with a supplement on the stability of steady motion. Proc. Lond. Math. Soc. 1875;6:86-97.

[23] Santoprete M. Linear stability of the Lagrangian triangle solutions for quasihomogeneous potentials. Celest. Mech. Dyn. Astron. 2006;94:17-35.

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