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Convergence Results of Two Step Iteration Procedure with Errors for a Pair of Asymptotically Non-Expansive Mappings in the Intermediate Sense

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Authors' contributions

This work was carried out in collaboration between both authors. Author RPB designed the study, wrote the first draft of the manuscript and managed literature searches. Author AP managed the analyses of the study. Both authors read and approved the final manuscript.

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Abstract

In this paper, we study strong and weak convergence results of a two step iterative process with errors for a pair of asymptotically non-expansive mapping in the intermediate sense.Our results generalize the corresponding results due to Hou and Du [4] by taking the class of asymptotically non-expansive mapping in the intermediate sense.We have also studied weak and strong convergence results under specific conditions.

Keywords: Asymptotically non-expansive mapping in the intermediate sense; condition A'; semi-compact.

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1 Introduction and Preliminaries

Let C be a non empty subset of a Banach space E, and let $T : C \to C$ be a mapping. Then T is said to be non-expansive [1] if

$$\| Tx - Ty \| \leq \| x - y \|, \text{ for all } x, y \in C.$$

In 1972, Geobel and Kirk [2] introduced the class of asymptotically non-expansive mappings as:

A self mapping T is said to be asymptotically non-expansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\| T^n x - T^n y \| \leq k_n \| x - y \|, \text{ for all } n \geq 1 \text{ and } x, y \in C.$$

Clearly the class of asymptotically non-expansive includes the class of non-expansive mappings as a proper subclass.

In 1993, Bruck, Kuczumov and Reich [3] introduced the class of asymptotically non-expansive in the intermediate sense as:

A self mapping T is said to be asymptotically non-expansive in the intermediate sense provided T is uniformly continuous and satisfies the following inequality:

$$limsup_{n\to\infty}(\| T^n x - T^n y \| - \| x - y \|) \leq 0, \text{ for all } x, y \in C.$$

If we take $\xi_n = max\{0, sup_{x,y\in C}(\| T^n x - T^n y \| - \| x - y \|)\}$,
then $\xi_n \to 0$ as $n \to \infty$. Hence, we obtain

$$\| T^n x - T^n y \| \leq \| x - y \| + \xi_n, \text{ for all } n \geq 1,\ x, y \in C.$$

It follows that asymptotically non-expansive mappings in the intermediate sense is more general than that of the asymptotically non-expansive mappings.

Example 1.1. *Let $X = R$ be a normed linear space and $h \in (0, 1)$. For each $x \in X$, we define*

$$T(x) = \begin{cases} \frac{hx}{2}, if\ x \neq 0 \\ 0, if\ x = 0. \end{cases}$$

Then

$\| T^n - T^n y \| = \frac{h^n}{2^n} \| x - y \| \leq \| x - y \|$*, for all $x, y \in X, n \in N$.*

Then T is an asymptotically non-expansive mapping with the constant sequence 1. *Now*
$limsup_{n\to\infty}(\| T^n x - T^n y \| - \| x - y \|)$

$$= limsup_{n\to\infty}\{\frac{h^n}{2^n} \| x - y \| - \| x - y \|\} \leq 0.$$

because $lim_{n\to\infty} h^n = 0$ for all $x, y \in H, n \in N$. Hence T is asymptotically non-expansive mapping in the intermediate sense.

In this paper we shall consider the following iterative scheme with error given by Hou and Du [4],
For $x_0 \in C$,

$$x_n = a_n x_{n-1} + b_n T^n y_n + c_n S^n x_n + e_n u_n,$$
$$y_n = a'_n x_{n-1} + b'_n x_n + c'_n S^n x_{n-1} + d'_n T^n x_n + e'_n v_n (1.1)$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{e_n\}, \{a_n^{'}\}, \{b_n^{'}\}, \{c_n^{'}\}, \{d_n^{'}\}, \{e_n^{'}\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n + e_n = 1, a_n^{'} + b_n^{'} + c_n^{'} + d_n^{'} + e_n^{'} = 1$, and $T, S : C \to C$ are both non-linear mappings and $\{u_n\}, \{v_n\} \in C$.

Now we recall the following concepts:

A mapping T with domain $D(T)$ and range $R(T)$ is said to be demiclosed at a point $p \in E$ if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx = p$.

A mapping T is said to be semicompact if for any sequence $\{x_n\}_{n=1}^{\infty}$ in C such that $lim_{n\to\infty} \parallel x_n - Tx_n \parallel = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $u \in C$.

A Banach space E is said to satisfy Opial' s condition [5] if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, then

$liminf_{n\to\infty} \parallel x_n - x \parallel \leq liminf_{n\to\infty} \parallel x_n - y \parallel$, for all $y \in E, y \neq x$.

Let C be a subset of a Banach space E. Two mappings $S, T : C \to C$ are said to satisfy condition condition $(A^{'})[6] if there exists a non-decreasing function$ f: $[0, \infty) \to [0, \infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\frac{1}{2}(\parallel x - Tx \parallel + \parallel x - Sx \parallel) \geq f(d(x, F))$$

for all $x \in C$ where $d(x, F) = inf\{\parallel x - x^* \parallel : x^* \in F(T) \cap F(S)\}$.

Lemma 1.2. *[7]Let $\{\ a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$.*

1. *if $\sum_{n=1}^{\infty} b_n < \infty$, then $lim_{n\to\infty} a_n$ exists.*
2. *if $\sum_{n=1}^{\infty} b_n < \infty$, and $\{a_n\}$ has a subsequence converging to zero, then $lim_{n\to\infty} a_n = 0$.*

Lemma 1.3. *[1] Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a non-empty closed convex subset of E. Let $T : C \to C$ be a non-expansive mapping. Then $(I - T)$ is demiclosed with respect to zero.*

Lemma 1.4. *[8] Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n.Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that $limsup_{n\to\infty} \parallel x_n \parallel \leq \alpha$, $limsup_{n\to\infty} \parallel y_n \parallel \leq \alpha$, and $limsup_{n\to\infty} \parallel t_n x_n + (1 - t_n) y_n \parallel = \alpha$, holds for some $\alpha \geq 0$. Then $lim_{n\to\infty} \parallel x_n - y_n \parallel = 0$.*

2 Main Results

Lemma 2.1. *Let H be a non-empty convex subset of a uniformly convex Banach space X and let $T_1, T_2 : H \to H$ be asymptotically non-expansive mappings in the intermediate sense. Let $\{x_n\}$ be the sequence defined by (1.1) with the following conditions:*

1. *$a_n \to 0, e_n \to 0, a_n^{'} \to 0, b_n^{'} \to 0, e_n^{'} \to 0$, as $n \to \infty$;*
2. *$b_n, c_n, c_n^{'}, d_n^{'} \in [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$;*
3. *$c_n^{'}, d_n^{'} \leq \beta$ for some $\beta \in (0, 1)$.*

If $F = F(T_1) \cap F(T_2) \neq \phi$, then we have

1. *$lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$ and $\{x_n\}, \{T_1^n x_n\}, \{T_2^n x_n\}$ are all bounded.*
2. *$lim_{n\to\infty}\|x_n - T_1^n x_n\| = 0 = lim_{n\to\infty}\|x_n - T_2^n x_n\|$.*

Proof. For any $p \in F$ we have

$$\begin{aligned} \| x_n - p \| &= \| a_n x_{n-1} + b_n T^n y_n + c_n S^n x_n + e_n u_n - p \| \\ &= \| a_n(x_{n-1} - p) + b_n(T^n y_n - p) + c_n(S^n x_n - p) + e_n(u_n - p) \| \\ &\leq a_n \| x_{n-1} - p \| + b_n \| T^n y_n - p \| + c_n \| S^n x_n - p \| \\ &+ e_n \| u_n - p \| \\ &\leq a_n \| x_{n-1} - p \| + b_n \| y_n - p \| + b_n \xi_n^{'} + c_n \| x_n - p \| \\ &+ c_n \xi_n^{''} + e_n \| u_n - p \| \ (2.1) \end{aligned}$$

Now

$$\begin{aligned} \| y_n - p \| &= \| a_n^{'} x_{n-1} + b_n^{'} x_n + c_n^{'} S^n x_{n-1} + d_n^{'} T^n x_n + e_n^{'} v_n - p \| \\ &\leq a_n^{'} \| x_{n-1} - p \| + b_n^{'} \| x_n - p \| + c_n^{'} \| S^n x_{n-1} - p \| \\ &+ d_n^{'} \| T^n x_n - p \| + e_n^{'} \| v_n - p \| \\ &\leq a_n^{'} \| x_{n-1} - p \| + b_n^{'} \| x_n - p \| + c_n^{'} \| x_{n-1} - p \| \\ &+ c_n^{'} \xi_n^{''} + d_n^{'} \| x_n - p \| + d_n^{'} \xi_n^{'} + e_n^{'} \| v_n - p \| \\ &\leq [a_n^{'} + c_n^{'}] \| x_{n-1} - p \| + [b_n^{'} + d_n^{'}] \| x_n - p \| + c_n^{'} \xi_n^{''} \\ &+ d_n^{'} \xi_n^{'} + e_n^{'} \| v_n - p \| \ (2.2) \end{aligned}$$

Now from (2.1) and (2.2) we have

$$\begin{aligned} \| x_n - p \| &\leq a_n \| x_{n-1} - p \| + b_n [a_n^{'} + b_n^{'}] \| x_{n-1} - p \| \\ &+ b_n [b_n^{'} + d_n^{'}] \| x_n - p \| + b_n c_n^{'} \xi_n^{''} + b_n d_n^{'} \xi_n^{'} + b_n e_n^{'} \| v_n - p \| \\ &+ b_n^{'} \xi_n^{'} + c_n \xi_n^{''} + c_n \| x_n - p \| + e_n \| u_n - p \| \\ &\leq [a_n + b_n(a_n^{'} + b_n^{'})] \| x_{n-1} - p \| + b_n(b_n^{'} + d_n^{'}) \| x_n - p \| \\ &+ c_n \| x_n - p \| + e_n \| u_n - p \| + b_n e_n^{'} \| v_n - p \| \\ &+ [b_n c_n^{'} \xi_n^{''} + b_n d_n^{'} \xi_n^{'} + b_n \xi_n^{'} + c_n \xi_n^{''}] \\ &\leq [a_n + b_n(a_n^{'} + b_n^{'}) \| x_{n-1} - p \| + [b_n(b_n^{'} + d_n^{'}) + c_n] \| x_n - p \| \\ &+ e_n \| u_n - p \| + b_n e_n^{'} \| v_n - p \| \\ &+ [b_n c_n^{'} \xi_n^{''} + b_n d_n^{'} \xi_n^{'} + b_n \xi_n^{'} + c_n \xi_n^{''}] (2.3) \end{aligned}$$

From equation (2.2) and (2.3) we conclude that

$$\| x_n - p \| \leq \| x_{n-1} - p \| \ (2.4)$$

and hence by lemma (1.2), $lim_{n\to\infty} \| x_n - p \|$ exists for all $p \in F$.

Also $\{x_n\}$ is bounded and so $\{Sx_n\}$ and $\{Tx_n\}$ are both bounded and hence $\{S^n x_n\}$ and $\{T^n x_n\}$ are also bounded.

Now suppose that $lim_{n\to\infty} \| x_n - p \| = \beta$ for some $\beta > 0$.

By inequalities (2.4) and (2.3) we have,

$$limsup_{n\to\infty} \parallel y_n - p \parallel \leq \beta (2.5)$$

From the iterative process (1.1), we have

$$\begin{aligned} \parallel x_n - p \parallel &= \parallel b_n[T^n y_n - p + a_n(x_{n-1} - S^n x_n) + e_n(u_n - S^n s_n)] \\ &+ (1 - b_n)[S^n x_n - p + a_n(x_{n-1} - S^n x_n) + e_n(u_n - S^n s_n)] (2.6) \end{aligned}$$

Since $a_n \to 0, \{x_n\}, \{S^n x_n\}$ and $\{T^n x_n\}$ are bounded. It follows from lemma (1.4) that,

$$lim_{n\to\infty} \parallel T^n y_n - S^n x_n \parallel = 0 (2.7)$$

Now by inequality (2.1) and (2.4) we have,

$$\begin{aligned} \parallel x_n - p \parallel &\leq a_n \parallel x_{n-1} - p \parallel + b_n \parallel y_n - p \parallel + c_n \parallel x_n - p \parallel \\ &+ e_n \parallel u_n - p \parallel + b_n \xi_n^{'} + c_n \xi_n^{''} \end{aligned}$$

$$(1 - c_n) \parallel x_n - p \parallel - a_n \parallel x_{n-1} - p \parallel - e_n \parallel u_n - p \parallel - b_n \xi_n^{'} - c_n \xi_n^{''} \leq b_n \parallel y_n - p \parallel$$

Taking liminf on both sides in the above inequality we have

$$\beta \leq liminf_{n\to\infty} \parallel y_n - p \parallel \leq limsup_{n\to\infty} \parallel y_n - p \parallel \leq \beta$$

which yields $lim_{n\to\infty} \parallel y_n - p \parallel = \beta (2.8)$.

Also

$$\begin{aligned} \parallel y_n - p \parallel &= \parallel c_n^{'}[(S^n x_{n-1} - p) + a_n^{'}(x_{n-1} - T^n x_n) + e_n^{'}(v_n - T^n x_n)] \\ &+ \parallel (1 - c_n^{'})[(T^n x_{n-1} - p) + a_n^{'}(x_{n-1} - T^n x_n) + e_n^{'}(v_n - T^n x_n)] \end{aligned}$$

Since $a_n^{'} \to 0, e_n^{'} \to 0, \{x_n\}, \{S^n x_n\}$ and $\{T^n x_n\}$ are bounded.

It follows from lemma (1.4) that,

$$lim_{n\to\infty} \parallel S^n x_{n-1} - T^n x_n \parallel = 0 (2.9)$$

Now by iteration scheme (1.1) we have,

$$\begin{aligned} \parallel x_n - T^n y_n \parallel &= a_n \parallel x_{n-1} - T^n y_n \parallel + c_n \parallel S^n x_n - T^n y_n \parallel \\ &+ e_n \parallel u_n - T^n y_n \parallel \end{aligned}$$

Since $a_n \to 0, e_n \to \infty$ as $n \to \infty$ and using the inequality (2.9), the above inequality becomes

$$lim_{n\to\infty} \parallel x_n - T^n y_n \parallel = 0 \ (2.10)$$

From inequalities (2.7) and (2.10) we can write,

$$\begin{aligned} \parallel x_n - S^n x_n \parallel &\leq \parallel x_n - T^n y_n \parallel + \parallel T^n y_n - S^n x_n \parallel \\ &\to 0 as n \to \infty . (2.11) \end{aligned}$$

Also

$$\begin{aligned} \parallel y_n - x_n \parallel &= \parallel a_n^{'} x_{n-1} + b_n^{'} x_n + c_n^{'} S^n x_{n-1} + d_n^{'} T^n x_n + e_n^{'} v_n - x_n \parallel \\ &= \parallel a_n^{'} x_{n-1} + (1 - a_n^{'} - c_n^{'} - e_n^{'}) x_n + c_n^{'} S^n x_{n-1} \\ &+ d_n^{'} T^n x_n + e_n^{'} v_n - x_n \parallel \\ &\leq a_n^{'} \parallel x_{n-1} - x_n \parallel + c_n^{'} \parallel S^n x_{n-1} - x_n \parallel \\ &+ d_n^{'} \parallel T^n x_n - x_n \parallel + e_n^{'} \parallel v_n - x_n \parallel (2.12) \end{aligned}$$

Now

$$\begin{aligned} \| x_n - T^n x_n \| &\leq \| x_n - S^n x_n \| + \| S^n x_n - T^n y_n \| + \| T^n y_n - T^n x_n \| \\ &\leq \| x_n - S^n x_n \| + \| S^n x_n - T^n y_n \| + \| y_n - x_n \| + \xi_n^{'} \\ &\leq \| x_n - S^n x_n \| + \| S^n x_n - T^n y_n \| \\ &\leq a_n^{'} \| x_{n-1} - x_n \| + c_n^{'} \| S^n x_{n-1} - x_n \| \\ &+ d_n^{'} \| T^n x_n - x_n \| + e_n^{'} \| v_n - x_n \| + \xi_n^{'} \end{aligned}$$

As $a_n^{'} \to 0, e_n^{'} \to 0, \xi_n^{'} \to 0$, as $n \to \infty$ and using $(2.7), (2.11), (2.10)$ in the above inequality we have

$$lim_{n \to \infty} \| x_n - T^n x_n \| = 0$$

This completes the proof. □

Theorem 2.2. *Let E be a uniformly convex Banach space and C be a non-empty bounded subset of it. Let $S, T : E \to E$ be two asymptotically non-expansive mappings in the intermediate sense and $\{x_n\}$ be the sequence defined by the iterative procedure* (1.1).*If $F = F(T) \cap F(S) \neq \phi$ and if one of the mapping S and T is semi compact then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. Since one of the S and T is semi compact so by definition there exists a subsequence $\{y_{n_j}\}$ of $\{x_n\}$ such that such that such that such that $\{y_{n_j}\}$ converges strongly to $\gamma.C$ is closed hence $\gamma \in C$. Now continuity of S and T implies that

$$\| S^n y_{n_j} - S^n \gamma \| \to 0 \text{ and}$$

$$\| T^n y_{n_j} - T^n \gamma \| \to 0 \text{ as } n_j \to \infty$$

Now from lemma (2.1)

$\| T^n \gamma - \gamma \| = 0 \Rightarrow \| T\gamma - \gamma \| = 0$. Similarly $\| S\gamma - \gamma \| = 0$. Hence $\gamma \in F$. Also lemma (2.1) yields that $lim_{n \to \infty} \| x_n - p \|$ exists for all $\gamma \in F$. Therefore $\{x_n\}$ must itself converge to $\gamma \in F$. This completes the proof. □

Theorem 2.3. *et E be a uniformly convex Banach space and C be a non-empty bounded subset of it. Let $S, T : E \to E$ be two asymptotically non-expansive mappings in the intermediate sense and $\{x_n\}$ be the sequence defined by the iterative procedure* (1.1).*If $F = F(T) \cap F(S) \neq \phi$ and if mappings S and T satisfy the condition $A^{'}$ then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. From lemma (2.1) $lim_{n \to \infty} \| x_n - p \|$ exists for all $p \in F$.Let $lim_{n \to \infty} \| x_n - p \| = \beta$.
If $\beta = 0$ then the result holds obviously.So let $\beta \geq 0$. Now by lemma (2.1)

$$lim_{n \to \infty} \| x_n - T^n x_n \| = 0 = lim_{n \to \infty} \| x_n - S^n x_n \|$$

Now by equation (2.4)

$$\| x_n - p \| \leq \| x_{n-1} - p \|$$

and hence

$$inf_{p \in F} \| x_n - p \| \leq inf_{p \in F} \| x_{n-1} - p \|$$

$$\Rightarrow d(x_n, F) \leq d(x_{n-1}, F)$$

Hence from lemma (1.2) $lim_{n\to\infty}d(x_n, F)$ exists.Now by condition $A^{'}$ $lim_{n\to\infty}f(d(x_n, p)) = 0$ for all $p \in F$.

Now we take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\{y_j\} \subset F$ such that

$$\| x_{n_j} - y_j \| \leq 2^{-j}$$

Next we claim that $\{y_j\}$ is a cauchy sequence in F. Now

$$\begin{aligned} \| y_{j+1} - y_j \| &\leq \| y_{j+1} - x_{n_j} \| + \| x_{n_j} - y_j \| \\ &\leq 2^{-j} + 2^{-j} \\ &= 2^{-j+1} \end{aligned}$$

which shows that $\{y_j\}$ is a cauchy sequence and hence convergent. Let $y_j \to y \in F$. Again by lemma (2.1) $lim_{n\to\infty} \| x_n - y \| = 0$. Hence $x_n \to y \in F$. This completes the proof. □

Theorem 2.4. *Let E be a uniformly convex Banach space satisfying Opial's condition ans C be its non-empty convex subset. Let $S, T : C \to C$ be two asymptotically non-expansive mappings in the intermediate sense and $\{x_n\}$ be the sequence defined by the iterative procedure* (1.1).*If $F = F(T) \cap F(S) \neq \phi$ then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. By lemma (2.1) $lim_{n\to\infty}d(x_n, p)$ exists for $p \in F$. Since every bounded subset of a uniformly convex space is weakly compact hence there exists a subsequence $\{x_{n_j}\}$ of the bounded sequence $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to a point $\beta \in C$.Now from lemma (2.1)

$$lim_{n_j} \to \infty \| T^n x_{n_j} - x_{n_j} \| = 0$$

By lemma (1.3), $(I - T)$ is demiclosed and hence $q \in F(T)$. By the similar arguments $\beta \in F(S)$. Hence $\beta \in F(T) \cap F(S)$.

Uniqueness: Let if possible there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $\{x_{n_j}\}$ converges to point $\beta^* \in C$. Now by the above arguments we have $\beta^* \in F(T) \cap F(S)$. By lemma (2.1) $lim_{n\to\infty} \| x_n - \beta \|$ and $lim_{n\to\infty} \| x_n - \beta^* \|$ exists.

Since E satisfies Opial condition, therefore

$$lim_{n_j\to\infty} \| x_{n_j} - \beta \| \leq lim_{n_j\to\infty} \| x_{n_j} - \beta^* \| \quad (2.13)$$

$$lim_{n_j\to\infty} \| x_{n_j} - \beta^* \| \leq lim_{n_j\to\infty} \| x_{n_j} - \beta \| \quad (2.14)$$

By (2.13) and (2.14) we have $\beta = \beta^*$.

Hence $\{x_n\}$ converges weakly to a common fixed point of S and T. □

3 Conclusion

The class of mapping used in this article is more general than that of non-expansive and asymptotically non-expansive mappings. Therefore the fixed point results derived by us are generalization of the previous results given in the existing literature.

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Competing Interests

Authors have declared that no competing interests exist.

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