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Boundary Layer Analysis of Unsteady Forced Convection of a Newtonian Fluid with Variable Thermo-physical Properties in the Presence of Induced Magnetic Field

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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Abstract

This paper attempts to effectively model the effects of variable viscosity and thermal conductivity on the unsteady hydromagnetic boundary layer flow past a semi-infinite plate when the oncoming free-stream is perturbed by an arbitrary function of time and applied magnetic field is far from and parallel to the plate. The two dimensional boundary layer equations are separated into those representing steady and unsteady parts of the flow. For, steady flow equation and unsteady flow equation, viscosity and thermal conductivity are considered as inverse linear functions of temperature. The basic steady flow governing partial differential equations are transformed into ordinary differential equations by means of similarity transformation which are solved numerically using shooting method and the resulting approximate solution have been used in the subsequent study of the unsteady flow. The unsteady flow equations are subject to the Laplace Transformation technique. In this case, solution for large time is obtained assuming velocity, temperature and Magnetic field as asymptotic expansion. The relevant flow and heat transfer characteristics that is the skin-friction coefficient, the plate temperature and the tangential magnetic field at the plate are derived and discussed numerically.

Keywords: MDH; variable viscosity; variable thermal conductivity.

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1 Introduction

The boundary layer flow of an electrically conducting fluid in the presence of magnetic field has wide applications in many engineering problems such as MHD generator, plasma studies, nuclear reactors, geothermal energy extraction, and oil exploration [1]. The problem of the boundary layer flow of a Newtonian fluid past a semi-infinite flat plate was first considered by Blasius [2]. Falkner and Skan [3] studied some approximate solution of the boundary layer flow problem past a semi-infinite plate. Again unsteady flows, such as start-up process and periodic fluid motion, are very much important in engineering practices [1]. Lighthill [4] had initiated the study of unsteady two dimensional boundary layer equations when the external flow fluctuates about a steady mean.

The study of hydromagnetic flow has stimulated considerable interest due to its important applications in cosmic-fluid dynamics, meteorology and solar physics and in the motion of the earth's core. Carrier and Greenspan [5] had studied the unsteady hydromagnetic boundary layer equations by using Osseen technique, when a semi-infinite plate moves impulsively in its own plane. Das [6] studied the unsteady hydomagnetic boundary layer flow past a semi infinite flat plate when the oncoming free-stream is perturbed by an arbitrary function of time and the applied magnetic field is parallel to the plate far away from it. Meksyn [7] considered the problem of flow of viscous electrically conducting fluid past a semi-infinite plate in the presence of an aligned magnetic field. Ingham [8] investigated the solution of the motion of a viscous electrically conducting fluid past a semi-infinite flat plate, which is started impulsively from rest with a constant velocity parallel to itself, in the presence of an applied magnetic field which is parallel to the plate at infinity. Very recently, Animasaun et al. [9,10] studied stagnation flow of nanofluid and unequal diffusivities case of homogeneous–heterogeneous reactions within viscoelastic fluid flow in the presence of induced magnetic-field.

Most of the existing analytical studies for this problem are based on the constant physical properties of the ambient fluid. However, it is known that these properties may change with temperature [11]. To accurately predict the flow and heat transfer rates it is necessary to take into account this variation of viscosity and thermal conductivity. A number of authors analysed the influence of variable thermo-physical properties on the flow structure and heat transfer. Mukhopadhyay [12] studied the effects of variable viscosity on the MHD boundary layer flow over a heated stretching surface. Sarma and Hazarika [13] studied on the effects of variable viscosity and thermal conductivity on heat and mass transfer flow along a vertical plate in the presence of a magnetic field. Salem [14] investigated variable viscosity and thermal conductivity effects on MHD flow and heat transfer in viscoelastic fluid over a stretching sheet. Seddeek et al. [15] studied the effects of variable viscosity and thermal conductivity on an unsteady two-dimensional laminar flow of a viscous incompressible conducting fluid past a semi-infinite vertical porous moving plate taking into account the effect of a magnetic field in the presence of variable suction. Odda and Farhan [16] have considered the effects of variable viscosity and variable thermal conductivity on heat transfer from a stretching sheet. The fluid viscosity and the thermal conductivity are assumed to vary as inverse linear functions of temperature. Pantokratoras [17] presented a theoretical study of the effects of variable fluid properties on the classical Blasius and Sakiadis flow. It is found that the variation of fluid properties and especially viscosity have a strong influence on the results.

The aim of this study is to investigate the effects of variable viscosity and thermal conductivity on the unsteady hydromagnetic boundary layer flow past a semi-infinite plate when the oncoming free-stream is perturbed by an arbitrary function of time and applied magnetic field is far from and parallel to the plate. The two dimensional boundary layer equations are separated into those representing steady and unsteady parts of the flow. For, steady flow equation and unsteady flow equation, viscosity and thermal conductivity are considered as inverse linear functions of temperature. The basic steady flow governing partial differential equations are transformed into ordinary differential equations by means of similarity transformation which are solved numerically using shooting method and the resulting approximate solution have been used in the subsequent study of the unsteady flow. The unsteady flow equations are subject to the Laplace Transformation technique. In this case, solution for large time is obtained assuming velocity, temperature and Magnetic field as asymptotic expansion. The relevant flow and heat transfer characteristics that is the skin-friction coefficient, the plate temperature and the tangential magnetic field at the plate are derived and discussed numerically.

2 Formulation of the Problem

Consider the unsteady flow of an incompressible electrically conducting fluid past a semi-infinite plate due to a steady magnetic field and an unsteady free-stream velocity both being parallel to the plate. The fluid properties are assumed to be isotropic and constant except for the fluid viscosity and thermal conductivity.

Let x and y be the distances measured along and perpendicular to the plate, (u,v) and (H_1, H_2) be the corresponding velocity and magnetic field components, U and H are the velocity and magnetic field at large distances from the plate and parallel to the plate. Both are assumed to be uniform.

The corresponding equations for the unsteady flow by making the usual boundary layer approximations

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\mu_e}{\rho_\infty}H\frac{\partial H}{\partial x}\\ +\frac{\mu_e}{\rho_\infty}\left(H_1\frac{\partial H_1}{\partial x}+H_2\frac{\partial H_1}{\partial y}\right)\end{aligned} \tag{2.1}$$

$$\frac{\partial H}{\partial t}+u\frac{\partial H_1}{\partial x}+v\frac{\partial H_1}{\partial y}-H_1\frac{\partial u}{\partial x}-H_2\frac{\partial u}{\partial y}=\eta_1\frac{\partial^2 H_1}{\partial y^2} \tag{2.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.3}$$

$$\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0 \tag{2.4}$$

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\\ \frac{1}{p_\infty c_p}\left[\left(E_z+\mu_e\left(uH_2-vH_2\right)\right)\left(\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}\right)\right]\end{aligned} \tag{2.5}$$

where ρ_∞ is the density of the fluid at infinity, μ is the viscosity of the fluid, k is the thermal conductivity, E_z is the z-component of electric field strength, $\eta_1=\dfrac{1}{\sigma\mu_e}$ is the magnetic diffusivity.

Also it is assumed that both the viscous Reynold number $R_a=\dfrac{U_0 x}{\vartheta_\infty}$ and the magnetic Reynold number $R_m=\dfrac{U_0 x}{\eta_1}$, U_0 being typical u velocity, are large compared with unity where $\vartheta_\infty=\dfrac{\mu_\infty}{\rho_\infty}$, μ_∞ being the viscosity of the fluid at infinity.

The normal component of the magnetic field is assumed to vanish at the wall while the parallel component approaches its given value at the edge of the boundary layer. Again since the fluid is finitely conducting and the plate is non-conducting there should be no surface current sheet and hence the tangential component of the magnetic field is continuous across the interface. The condition is expressed by the equation $\frac{\partial H_1}{\partial y} = 0$ at y =0.

Hence the boundary conditions are

$$\left.\begin{array}{l} u = 0 = v,\ \frac{\partial H_1}{\partial y} = 0,\ H_2 = 0,\ T = T_w \text{ at } y = 0 \\ u \to U(t),\ \ H_1 = H,\ \ T \to T_\infty \text{ as } y \to \infty \end{array}\right\} \tag{2.6}$$

The fluid viscosity is assumed to be an inverse linear function of temperature [18],

$$\frac{1}{\mu} = \frac{1}{\mu_\infty}\left[1 + \gamma(T - T_\infty)\right] = a(T - T_r) \tag{2.7}$$

$$a = \frac{\gamma}{\mu_\infty}, T_r = T_\infty - \frac{1}{\gamma}$$

a and T_r are constants and their values depend on the reference state and thermal property of the fluid. In general $a > 0$ for liquids and $a < 0$ for gases. γ is a constant based on thermal property of the fluid. For $\gamma \to 0$, $\mu = \mu_\infty$ (constant).

The thermal conductivity is also assumed to be an inverse linear function of temperature [19],

$$\frac{1}{k} = \frac{1}{k_\infty}[1 + \xi(T - T_\infty) = c(T - T_k)] \tag{2.8}$$

$$c = \frac{\xi}{k_\infty}, T_k = T_\infty - \frac{1}{\xi}$$

Where c and T_k are constants and their values depend on the reference state and thermal property of the fluid. ξ is a constant based on thermal property of the fluid. c>0 for liquids and c<0 for gases.

Following Lighthill [8], we have

$$\left.\begin{array}{l} u = u_o(x, y) + \varepsilon\, u_1(x, y, t) \\ v = v_o(x, y) + \varepsilon\, v_1(x, y, t) \\ U(t) = U_o + \varepsilon\, U_1(t) \\ H_1 = H_{xo}(x, y) + \varepsilon\, H_{x1}(x, y, t) \\ H_2 = H_{yo}(x, y) + \varepsilon\, H_{y1}(x, y, t \\ H = H_o \\ T = T_o(x, y) + \varepsilon\, T_1(x, y, t) \end{array}\right\} \tag{2.9}$$

where the o subscript quantities denote the steady motion, and 1 subscript quantities denote the unsteady motion, ε being a small reference parameter.

Substituting (2.7) in equations (2.1) - (2.5) and equating coefficient of 0(ε), equations for the steady state,

$$u_o \frac{\partial u_o}{\partial x} + v_o \frac{\partial u_o}{\partial y} = \frac{1}{\rho_\infty} \frac{\partial}{\partial y}\left(\mu \frac{\partial u_o}{\partial y} \right) + \frac{\mu_e}{\rho_\infty}\left\{ H_{xo} \frac{\partial H_{xo}}{\partial x} + H_{y0} \frac{\partial H_{xo}}{\partial y} \right\} \tag{2.10}$$

$$u_o \frac{\partial H_{x_o}}{\partial x} + v_o \frac{\partial H_{x_o}}{\partial y} - H_{x_o} \frac{\partial u_o}{\partial x} - H_{y0} \frac{\partial u_o}{\partial x} = \eta_1 \frac{\partial^2 H_{x_o}}{\partial y^2} \tag{2.11}$$

$$\frac{\partial u_o}{\partial x} + \frac{\partial v_o}{\partial y} = 0 \tag{2.12}$$

$$\frac{\partial H_{x_o}}{\partial x} + \frac{\partial H_{y_o}}{\partial y} = 0 \tag{2.13}$$

$$\begin{aligned} u_o \frac{\partial T_o}{\partial x} + v_o \frac{\partial T_o}{\partial y} &= \frac{1}{\rho_\infty c_p} \frac{\partial}{\partial y}\left(k \frac{\partial T_o}{\partial y} \right) \\ &\quad + \frac{1}{\rho_\infty c_p}\left[\left\{ E_z + \mu_e \left(u_o H_{y_o} - v_0 H_{x_o} \right) \right\} \left(\frac{\partial H_{y_o}}{\partial x} - \frac{\partial H_{x_o}}{\partial y} \right) \right] \end{aligned} \tag{2.14}$$

The corresponding boundary conditions are

$$\left. \begin{aligned} & u_o = 0,\ v_o = 0,\ \frac{\partial H_{xo}}{\partial y} = 0,\ H_{yo} = 0,\ T_o = T_{ow} \ \text{ at } \ y=0 \\ & u_o = U_o,\ H_{xo} = H_o,\ T_o \to T_\infty \qquad \text{as } \ y \to \infty \end{aligned} \right\} \tag{2.15}$$

The equations for unsteady state

$$\begin{aligned} & \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_o}{\partial x} + u_o \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_o}{\partial y} + v_o \frac{\partial u_1}{\partial y} = \frac{\partial U_1}{\partial t} + \frac{1}{\rho_\infty} \frac{\partial}{\partial y}\left(\mu \frac{\partial u_1}{\partial y} \right) \\ & \quad + \frac{\mu_e}{\rho_\infty}\left\{ H_{xo} \frac{\partial H_{x1}}{\partial x} + H_{x1} \frac{\partial H_{xo}}{\partial x} + H_{yo} \frac{\partial H_{x1}}{\partial y} + H_{y1} \frac{\partial H_{xo}}{\partial y} \right\} \end{aligned} \tag{2.16}$$

$$\begin{aligned} & \frac{\partial H_{x1}}{\partial t} + u_o \frac{\partial H_{x1}}{\partial x} + u_1 \frac{\partial H_{xo}}{\partial x} + v_o \frac{\partial H_{x1}}{\partial y} + v_1 \frac{\partial H_{xo}}{\partial y} - H_{xo} \frac{\partial u_1}{\partial x} \\ & \qquad - H_{x1} \frac{\partial u_o}{\partial x} - H_{yo} \frac{\partial u_1}{\partial y} - H_{y1} \frac{\partial u_o}{\partial y} = \eta_1 \frac{\partial^2 u_1}{\partial y^2} \end{aligned} \tag{2.17}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{2.18}$$

$$\frac{\partial H_{x1}}{\partial x} + \frac{\partial H_{y1}}{\partial y} = 0 \tag{2.19}$$

$$\frac{\partial T_1}{\partial t}+u_o\frac{\partial T_1}{\partial x}+u_1\frac{\partial T_o}{\partial x}+v_1\frac{\partial T_o}{\partial y}+v_o\frac{\partial T_1}{\partial y}=\frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(k\frac{\partial T_1}{\partial y}\right)+$$
$$\frac{1}{\rho_\infty c_p}\left[\begin{array}{l}\left\{E_z+\mu_e\left(u_o H_{yo}-v_o H_{xo}\right)\right\}\left\{\dfrac{\partial H_{y1}}{\partial x}-\dfrac{\partial H_{x1}}{\partial y}\right\}+\\ \mu_e\left\{\left(u_o H_{y1}+u_1 H_{yo}\right)-\left(v_o H_{x1}+v_1 H_{xo}\right)\right\}\left\{\dfrac{\partial H_{yo}}{\partial x}-\dfrac{\partial H_{xo}}{\partial y}\right\}\end{array}\right] \tag{2.20}$$

The corresponding boundary conditions are

$$\left.\begin{array}{l}u_1=0=v_1,\quad \dfrac{\partial H_{x1}}{\partial y}=0,\quad \partial H_{y1}=0,\quad \dfrac{\partial T_1}{\partial y}=0 \quad \text{at } y=0\\ u_1=U_1(t)=U_0 U_M(t),\ H_{x1}\to 0,\ T_1\to T_\infty \quad \text{as } y\to\infty\end{array}\right\} \tag{2.21}$$

where $U_M(t)$ is an arbitrary function of time.

3 Steady State Solution

Equations (2.10) - (2.14) together with boundary conditions (2.15) are the magnetohydrodynamic boundary layer equations for a steady flow of an electrically conducting fluid past a semi-infinite non-conducting plate. Using the non-dimensional transformations

$$\left.\begin{array}{l}\eta=\sqrt{\dfrac{U_o}{\vartheta_\infty x}}y\\ u_o=U_o f'(\eta),\qquad v_o=\dfrac{1}{2}\sqrt{\dfrac{\vartheta_\infty U_o}{x}}\left(\eta f'-f\right)\\ H_{xo}=H_o g'(\eta),\quad H_{yo}=\dfrac{1}{2}H_o\sqrt{\dfrac{\vartheta_\infty}{U_o x}}\left(\eta g'-g\right)\\ T_o=T_\infty+\left(T_{ow}-T_\infty\right)\theta_o(\eta)\end{array}\right\} \tag{3.1}$$

in equations (2.10) - (2.14), the equation of continuity for velocity and magnetic field are satisfied automatically. The equation of momentum, the magnetic induction equation and the heat transfer equation become

$$f'''-\frac{\theta_o' f''}{\theta_o-\theta_{ro}}+\frac{\theta_{ro}-\theta_o}{2\theta_{ro}}\left(ff''-R_H gg''\right)=0 \tag{3.2}$$

$$\frac{2}{P_m}g'''=gf''-fg'' \tag{3.3}$$

$$\frac{1}{P_r}\left(\theta_o''-\frac{\theta_o'^2}{\theta_o-\theta_{ko}}\right)+\frac{1}{2}f\theta'+\frac{E_c R_H}{R_a}\left\{R_e\sqrt{R_a}+\frac{1}{2}\left(fg'-gf'\right)\right\}\times\left\{\frac{1}{4}\left(g-\eta g'-\eta^2 g''\right)-R_a g''\right\}=0 \tag{3.4}$$

with the boundary conditions

$$\left.\begin{array}{l} f' = 0 = f,\;\; g'' = 0 = g,\; \theta_0 = 1 \text{ at } \eta=0 \\ f' = 1,\; g' = 1,\; \theta_0 = 0 \text{ as } \eta\to\infty \end{array}\right\} \tag{3.5}$$

Where $P_r = \dfrac{\vartheta_\infty}{K}$ is the Prandtl number, $P_m = \dfrac{\vartheta_\infty}{\eta_1}$ is the magnetic Prandtl number, $R_H = \dfrac{\mu_e H_0^2}{\rho_\infty U_0^2}$ is the magnetic pressure number, $R_e = \dfrac{E_z}{\mu_e U_0 H_0}$ is the electric field parameter, $E_c = \dfrac{U_0^2}{c_p(T_{0w} - T_\infty)}$ is the Eckert number, $\theta_{r0} = \dfrac{T_r - T_\infty}{T_{0w} - T_\infty}$ is the viscosity variation parameter, $\theta_{k0} = \dfrac{T_k - T_\infty}{T_{0w} - T_\infty}$ is the thermal conductivity variation parameter.

The physical quantities of interest are the local Skin-friction coefficient C_f, surface magnetic field H_{xo} and the local Nusselt number *Nu.*

The Skin Friction Coefficient at the plate is given by

$$C_f = \frac{2\tau_w}{\rho_\infty {U_0}^2} = 2\frac{\theta_{r0}}{\theta_{r0} - 1}(R_a)^{-1/2} f''(0) \tag{3.6}$$

where

$$\begin{aligned} \tau_w &= \mu\left(\frac{\partial u_0}{\partial y}\right)_{y=0} \\ &= \mu_\infty \frac{\theta_{r0}}{\theta_{r0} - 1} U_0 \sqrt{\frac{U_0}{\vartheta_\infty x}} f''(0) \end{aligned}$$

is the tangential shear stress.

The tangential component of the magnetic field at the plate is given by

$$H_{xo}(0) = H_o g'(0) \tag{3.7}$$

The Nusselt number at the plate is given by

$$\begin{aligned} Nu &= \frac{x q_w}{k_\infty (T_{0w} - T_\infty)} \\ &= -\frac{\theta_{k0}}{\theta_{k0} - 1} (R_a)^{1/2} \theta'(0) \end{aligned} \tag{3.8}$$

where $q_w = -k\left(\dfrac{\partial T_0}{\partial y}\right)_{y=0}$ is the heat flux at the wall.

4 Unsteady State Solutions

Substituting $t_1 = ct,\ c > 0$ and multiplying equations (2.16) - (2.20) by e^{-st_1} and integrating with respect to t_1 from 0 to ∞ (*i.e.* applying Laplace transformation) [5], we have

$$cs\bar{u}_1 + u_0\frac{\partial \bar{u}_1}{\partial x} + \bar{u}_1\frac{\partial u_0}{\partial x} + v_0\frac{\partial \bar{u}_1}{\partial y} + \bar{v}_1\frac{\partial u_0}{\partial y} = cs\bar{U}_1 + \frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}_1}{\partial y}\right) + \frac{\mu_e}{\rho_\infty}\begin{bmatrix} H_{x0}\dfrac{\partial \bar{H}_{x1}}{\partial x} + \bar{H}_{x1}\dfrac{\partial \bar{H}_{x0}}{\partial x} \\ + H_{y0}\dfrac{\partial \bar{H}_{x1}}{\partial y} + \bar{H}_{y1}\dfrac{\partial \bar{H}_{x0}}{\partial y} \end{bmatrix} \tag{4.1}$$

$$cs\bar{H}_{x1} + u_0\frac{\partial \bar{H}_{x1}}{\partial x} + \bar{u}_1\frac{\partial \bar{H}_{x0}}{\partial x} + v_0\frac{\partial \bar{H}_{x1}}{\partial y} + \bar{v}_1\frac{\partial \bar{H}_{x0}}{\partial y} - H_{x0}\frac{\partial \bar{u}_1}{\partial x} - \bar{H}_{x1}\frac{\partial \bar{u}_0}{\partial x} - H_{y0}\frac{\partial \bar{u}_1}{\partial y} - \bar{H}_{y1}\frac{\partial \bar{u}_0}{\partial y} = \mu_1\frac{\partial^2 \bar{H}_{x1}}{\partial y^2} \tag{4.2}$$

$$\frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{v}_1}{\partial y} = 0 \tag{4.3}$$

$$\frac{\partial \bar{H}_{x1}}{\partial x} + \frac{\partial \bar{H}_{y1}}{\partial y} = 0 \tag{4.4}$$

$$cs\bar{T}_1 + u_0\frac{\partial \bar{T}_1}{\partial x} + \bar{u}_1\frac{\partial T_0}{\partial x} + v_0\frac{\partial \bar{T}_1}{\partial y} + \bar{v}_1\frac{\partial T_0}{\partial y} = \frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}_1}{\partial y}\right) + \frac{1}{\rho_\infty c_p}\begin{bmatrix} \{E_z + \mu_e(u_0H_{y0} - v_0H_{x0})\}\left\{\dfrac{\partial \bar{H}_{y1}}{\partial x} - \dfrac{\partial \bar{H}_{x1}}{\partial y}\right\} \\ + \mu_e\{(u_0\bar{H}_{y1} + \bar{u}_1H_{y0}) - (v_0\bar{H}_{x1} + v_1H_{x0})\} \\ \times\left\{\dfrac{\partial H_{y0}}{\partial x} - \dfrac{\partial H_{x0}}{\partial y}\right\} \end{bmatrix} \tag{4.5}$$

where $\bar{u}_1 = \int_0^\infty u_1e^{-st_1}dt_1$, Rl (s) > 0, etc.

Also we assumed that

$u_1e^{-st_1} \to 0,\ U_1e^{-st_1} \to 0$ as $t_1 \to \infty$,
u_1=0, U_1=0 at t_1=0,
$H_{x1}e^{-st_1} \to 0$ as $t_1 \to \infty$,
H_{x1}= 0 at $t_1 = 0$,

$T_1 e^{-st_1} \to 0$ as $t_1 \to \infty$,

and $T_1 = 0$ at $t_1 = 0$.

The transformed boundary conditions are

$$\left.\begin{aligned} &\bar{u}_1 = 0 = \bar{v}_1,\ \frac{\partial \bar{H}_{x1}}{\partial y} = 0,\ \bar{H}_{y1} = 0,\ \frac{\partial \bar{T}_1}{\partial y} = 0 \quad \text{at } y=0 \\ &u_1 \to \bar{U}(s) = U_0 \bar{U}_M(s),\ \bar{H}_{x1} = 0,\ \bar{T}_1 \to T_\infty \quad \text{as } y \to \infty \end{aligned}\right\} \tag{4.6}$$

$\bar{U}_M(s)$ being the transformation of $U_M(t)$.

4.1 Solution for large time

In the *s*-plane large time will correspond to small values of *s*. The assumption $U_1(t) = U_0 U_M(t)$ in the boundary conditions (2.21) implies that the main stream has been perturbed only in magnitude but not in direction. To solve equations (4.1) - (4.5) we expand $\bar{u}_1,\ \bar{v}_1,\ \bar{H}_{x1},\ \bar{H}_{y1},\ \bar{T}_1$ in series of ascending powers of *s* as follows (Das [6]),

$$\left.\begin{aligned} &\bar{u}_1 = U_0 \bar{U}_M(s) \sum_{n=0}^{\infty} s^n x^n F_n'(\eta) \\ &\bar{v}_1 = \sqrt{U_0 \vartheta_\infty}\, \bar{U}_M(s) \sum_{n=0}^{\infty} s^n x^{n-\frac{1}{2}} \left[\frac{1}{2}\eta F_n'(\eta) - \left(n+\frac{1}{2}\right) F_n(\eta)\right] \\ &\bar{H}_{x1} = H_0 \bar{U}_M(s) \sum_{n=0}^{\infty} s^n x^n G_n'(\eta) \\ &\bar{H}_{y1} = H_0 \sqrt{\frac{\vartheta_\infty}{U_0}}\, \bar{U}_M(s) \sum_{n=0}^{\infty} s^n x^{n-\frac{1}{2}} \left[\frac{1}{2}\eta G_n'(\eta) - \left(n+\frac{1}{2}\right) G_n(\eta)\right] \\ &\bar{T}_1 = T_\infty + (T_{0w} - T_\infty) \bar{U}_M(s) \sum_{n=0}^{\infty} s^n x^n \psi_n(\eta) \end{aligned}\right\} \tag{4.7}$$

Substituting for $u_0, v_0, H_{x0}, H_{y0}, T_0$ from (3.1) and for $\bar{u}_1, \bar{v}_1, \bar{H}_{x1}, \bar{H}_{y1}, \bar{T}_1$ from (4.7) in equations (4.1) - (4.5), the equations of continuity for velocity and magnetic field are satisfied automatically and from the remaining three equations after comparing the coefficients of s^n we have, for n=0

$$\frac{2\theta_{r0}}{\theta_0 - \theta_{r0}}\left(F_0''' - \frac{\theta_0' F_0''}{\theta_0 - \theta_{r0}}\right) + F_0'' f + F_0 f'' - R_H\left(g G_0'' + g'' G_0\right) = 0 \tag{4.8}$$

$$\frac{2}{P_m} G_0''' = f'' G_0 + g F_0'' - f G_0'' - g'' F_0 \tag{4.9}$$

$$\begin{aligned} &\frac{1}{P_r}\left(\psi_0'' - \frac{\theta_0' \psi_0'}{\theta_0 - \theta_{k0}}\right) + \frac{1}{2}\left(f\psi_0' + F_0\theta_0'\right) - \frac{E_c R_H}{4R_a}\Big[\left\{R_e\sqrt{R_a} + \frac{1}{2}\left(fg' - f'g\right)\right\} \\ &\left\{\eta^2 g'' + 4R_a G_0''\right\} - \frac{1}{2}\left\{G_0' f - G_0 f' + F_0 g' - F_0' g\right\}\left\{\left(\eta^2 g'' + \eta g' - g\right) - 4R_a g''\right\}\Big] = 0 \end{aligned} \tag{4.10}$$

The boundary conditions are

$$\left.\begin{array}{l}\eta=0\ ;\ F_0''=0=F_0\ ,\quad G_0''=0=G_0\ ,\ \psi_0'=0\\ \eta\to\infty\ ;\ F_0'=1,\ G_0'=0,\quad \psi_0=0\end{array}\right\}\tag{4.11}$$

For n=1,

$$\frac{2\theta_{r0}}{\theta_0-\theta_{r0}}\left(F_1'''-\frac{\theta_0' F_1''}{\theta_0-\theta_{r0}}\right)+fF_1''-2f'F_1'+ \\ 3f''F_1-2F_0'+2+R_H\left(2G_1'g'-gG_1''-3g''G_1\right)=0\tag{4.12}$$

$$\frac{1}{P_m}G_1'''=G_0'+\frac{1}{2}\left(3f''G_1+2f'g_1'-fG_1''\right) \\ -\frac{1}{2}\left(3g''F_1+2g'F_1'-gF_1''\right)\tag{4.13}$$

$$\frac{2}{P_r}\left(\psi_1''-\frac{\theta_0'\psi_1'}{\theta_0-\theta_{k0}}\right)+3F_1\theta_0'+f\psi_1'-2f'\psi_1-2\psi_0 \\ -\frac{E_cR_H}{4R_a}\left[\begin{array}{l}\left\{R_e\sqrt{R_a}+\frac{1}{2}\left(fg'-f'g\right)\right\}\left\{6G_1+\eta^2G_1''+4R_aG_1''\right\}\\ -\frac{1}{2}\left(fG_1'-3f'G_1+3F_1g'-F_1'g\right)\left\{\begin{array}{l}\left(\eta^2g''+\eta g'-g\right)\\ -4R_ag''\end{array}\right\}\end{array}\right]=0\tag{4.14}$$

The boundary conditions are

$$\left.\begin{array}{l}\eta=0;\quad F_1'=0=F_1,\quad G_1''=0=G_1,\ \psi_1'=0\\ \eta\to\infty;\ F_1'=1,\ G_1'=0,\quad \psi_1=0\end{array}\right\}\tag{4.15}$$

Followind Das [6], let

$$\bar{u}_1=\bar{u}_2(x,y,s)\bar{U}_M(s)\tag{4.16}$$

So that $\bar{u}_2=U_0\sum_{n=0}^{\infty}s^nx^nF_n'(\eta)$ (4.17)

Inverting (4.16) [6],

$$u_1=\int_0^{t_1}U_M(t_1-\tau)u_2(x,y,\tau)d\tau\tag{4.18}$$

And inverting (4.17) term by term

$$u_2 = U_0 \sum_{n=0}^{\infty} F_n'(\eta) \frac{d^n \partial(t_1)}{dt_1^n}$$ where $\partial(t_1)$ is the delta function.

Therefore from (4.18)

$$\begin{aligned} u_1 &= U_0 \sum_{n=0}^{\infty} x^n F_n'(\eta) \int_0^{t_1} U_M (t_1 - \tau) \frac{d^n \partial(t_1)}{dt_1^n} d\tau \\ &= U_0 \sum_{n=0}^{\infty} x^n F_n'(\eta) \frac{d^n U_M}{dt_1^n} \end{aligned} \tag{4.19}$$

similarly

$$H_{x1} = H_0 \sum_{n=0}^{\infty} x^n G_n'(\eta) \frac{d^n U_M}{dt_1^n} \tag{4.20}$$

and

$$T_1 = T_\infty + (T_{w0} - T_\infty) \sum_{n=0}^{\infty} x^n \psi_n(\eta) \frac{d^n U_M}{dt_1^n} \tag{4.21}$$

The equations (4.19), (4.20) and (4.21) give the unsteady part of velocity field, magnetic field and temperature field for large time.

The unsteady part of the Skin-friction C_{1f}, the surface magnetic field at the plate H_{x1} and the plate temperature T_1 are given by

$$\begin{aligned} C_{1f} &= \frac{2\tau_{1w}}{\rho_\infty U_0^2} \\ &= 2 \frac{\theta_{r0}}{\theta_{r0} - 1} (R_a)^{-1/2} \sum_{n=0}^{\infty} x^n F_n''(0) \frac{d^n U_M}{dt_1^n} \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \tau_{1w} &= \mu_\infty \left(\frac{\partial u_1}{\partial y} \right)_{y=0} \\ &= \mu_\infty U_0 \sqrt{\frac{U_0}{\mu_\infty x}} \sum_{n=0}^{\infty} x^n F_n''(0) \frac{d^n U_M}{dt_1^n} \end{aligned}$$

is the shear stress at the wall.

$$H_{x1} = H_0 \sum_{n=0}^{\infty} x^n G_n'(0) \frac{d^n U_M}{dt_1^n} \tag{4.23}$$

and

$$T_1 = T_\infty + \left(T_{w0} - T_\infty\right)\sum_{n=0}^{\infty} x^n \psi_n(0)\frac{d^n U_M}{dt_1^n} \tag{4.24}$$

5 Results and Discussion

The transformed dimensionless coupled non-linear equations (3.2)-(3.4) together with boundary conditions (3.5) for steady state and equations (4.8) - (4.11) and (4.12) - (4.15) for unsteady state are solved using shooting method for various combination of parameters viz. viscosity variation parameter θ_r, thermal conductivity variation parameter θ_k, magnetic pressure number R_H, magnetic Prandtl number P_m for Eckert number E_c=0.01, electric field parameter R_e=-1 and Reynolds number R_a=2 for fluid with Prandtl number P_r=0.7. The corresponding non-dimensional skin-friction coefficient $f''(0)$, the magnetic field at the plate $g'(0)$ and the heat transfer rate at the surface $\theta'(0)$ for steady state and the unsteady part of the Skin-friction $F_n''(0)$, the surface magnetic field at the plate $G_n'(0)$ and the plate temperature $\Psi_n(0)$ for n=0 and n=1 are evaluated and tabulated.

Table 1 contains numerical result for the skin-friction parameter $f''(0)$, $F_o''(0)$, $F_1''(0)$ at the plate for P_r=0.7, θ_k=-10, R_e=-1, E_c=0.01, R_H=0.5, P_m =0.5, R_a=2 for different values of θ_r varying from -10 to -1. The result show that $f''(0)$, $F_o''(0)$, increases but $F_1''(0)$ decreases when θ_r increases. This effects on $g'(0)$ and $\theta'(0)$ are less significant. The numerical values for the heat transfer rate at the surface $\theta'(0)$ for steady state are shown in Table 2. It is clear from this Table that the heat transfer rate at the surface for steady state decreases with the increasing values of thermal conductivity variation parameter θ_k. Same is the case with the unsteady part of plate temperature $\Psi_o(0)$ and $\Psi_1(0)$. These effects on $F_o''(0)$, $G_o'(0)$ are not significant. It is seen from Tables 3, 4 and 5 that the surface magnetic field $g'(0)$, $G_o'(0)$, $G_1'(0)$ increases as magnetic Prandtl number P_m decreases. $\Psi_1(0)$ decrease and $F_o''(0)$, $F_1''(0)$ increases with the increasing values of P_m. This is because as P_m decreases, the magnetic diffusivity $\eta_1 = \dfrac{1}{\sigma\mu_e}$ becomes larger resulting in greater horizontal component $g'(0)$ of the magnetic field near the wall. Again $f''(0)$ increases as P_m decreases because the conductivity σ is reduced and the boundary layer velocity begins to lose control over the magnetic lines of force. Consequently the induced normal component of the magnetic field decreases and along with it the ponder motive force (the $\mu_e\bar{J}\times\bar{H}$ force, $\bar{J}$ being current) which resist the fluid motion parallel to the plate is reduced. This tends to increase the skin-friction.

Fig. 1 depicts the influence of viscosity variation parameter θ_r on the flow field. The velocity boundary layer increases with increasing θ_r. Fig. 2 shows that the thermal boundary layer decreases for increasing values of thermal conductivity variation parameter θ_k.

Table 1. Values of Skin-friction $f''(0)$, $F_o''(0)$ and $F_1''(0)$ for different values of θ_r and for P_r=0.7, θ_k=-10, R_e=-1, E_c=0.01, R_H=0.5, P_m=0.5, R_a=2.

θ_r	$f''(0)$	$F_o''(0)$	$F_1''(0)$
-10	0.729577	0.464715	-0.44653
-8	0.737904	0.469192	-0.4545
-6	0.751394	0.476366	-0.46771
-4	0.777012	0.489708	-0.49384
-3	0.800973	0.501831	-0.51963
-1	0.949221	0.522924	-0.71849

Table 2. Values of the heat transfer $\theta'(0)$, $\Psi_0(0)$ and $\Psi_1(0)$ for different values of θ_k and for P_r=0.7, θ_r=-10, R_e=-1, E_c=0.01, R_H=0.5, P_m=0.5, R_a=2.

θ_k	$\theta'(0)$	$\Psi_o(0)$	$\Psi_1(0)$
-10	-0.71101	-0.07142	0.04392
-8	-0.7189	-0.07694	0.043204
-6	-0.73179	-0.0856	0.041991
-4	-0.75666	-0.10117	0.039509
-2	-0.82493	-0.13758	0.031887
-1	-0.94015	-0.18386	0.017105

Table 3. Values of Skin-friction $f''(0)$ and the magnetic field at the plate $g'(0)$ for different values of P_m, and for θ_r=-10, θ_k=-10, R_e=-1, E_c=0.01, R_H=0.2, P_r=0.7, R_a=2.

P_m	$f''(0)$	$g'(0)$
0.2	0.41396	0.899385
0.4	0.4115	0.821799
0.6	0.409834	0.760043
0.8	0.408662	0.709682
1	0.407814	0.66781
2	0.405813	0.532778
3	0.405148	0.459104
4	0.404857	0.412324

Table 4. Values of Skin-friction $F_o''(0)$, the magnetic field at the plate $G_o'(0)$ and plate temperature $\Psi_o(0)$ for different values of P_m. for θ_k=-10, θ_r=-10, R_e=-1, E_c=0.01, R_H=0.5, P_r=0.7, R_a=2

P_m	$F_o''(0)$	$G_o'(0)$	$\Psi_o(0)$
0.01	0.441083	0.994712	-0.07068
0.04	0.443116	0.979174	-0.07075
0.07	0.44505	0.96411	-0.07082
0.1	0.446891	0.9495	-0.07088
0.3	0.457152	0.862442	-0.07121
0.5	0.464715	0.790322	-0.07142
1	0.486049	0.498313	-0.07167
2	0.489234	0.412218	-0.07152

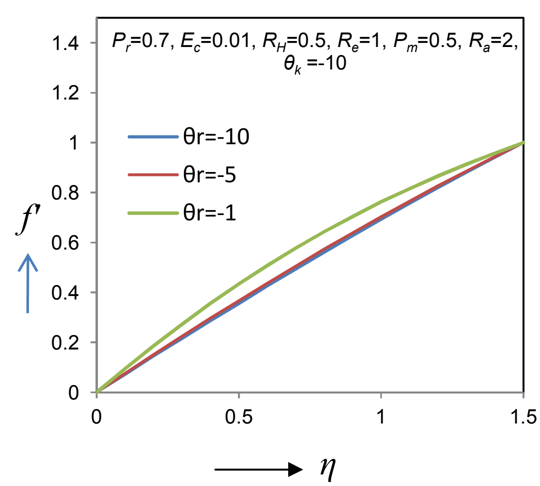

Fig. 1. Variation of velocity profiles for θ_r

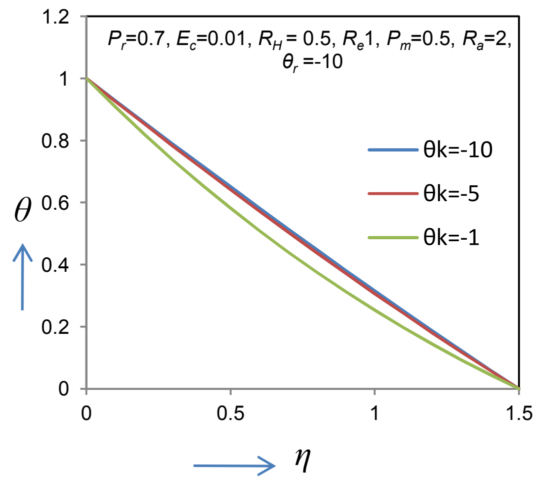

Fig. 2. Variation of temperature profiles for θ_k

Table 5. Values of Skin-friction $F_1''(0)$, the magnetic field at the plate $G_1'(0)$ and plate temperature $\Psi_1(0)$ for different values of P_m. for θ_k=-10, θ_r=-10, R_e=-1, E_c=0.01, R_H =0.5, R_a=2, P_r =0.7

P_m	$F_1''(0)$	$G_1'(0)$	$\Psi_1(0)$
0.01	-0.57628	-0.00888	0.054951
0.04	-0.56516	-0.03497	0.05399
0.07	-0.55457	-0.06022	0.053076
0.1	-0.54446	-0.08468	0.052206
0.3	-0.48797	-0.22976	0.047388
0.5	-0.44653	-0.34888	0.04392
1	-0.385	-0.5674	0.038997
2	-0.34826	-0.81341	0.036681

6 Conclusions

Under the assumption of temperature dependent viscosity and thermal conductivity the unsteady hydromagnetic boundary layer flow past a semi-infinite plate when the oncoming free-stream is perturbed by an arbitrary function of time is studied. The result pertaining to the present study indicate that the temperature dependent fluid viscosity and thermal conductivity play an important role in skin friction factor, surface magnetic field and plate temperature. The effects of magnetic parameters on the flow field are apparent.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Nandy SK, Sidui S, Mahapatra TR. Unsteady MHD boundary-layer flow and heat transfer of nanofluid over a permeable shrinking sheet in the presence of thermal radiation. Alexandria Engineering Journal. 2014;53:929–937.

[2] Blasius H, Math Physik Z. Bd. Engl. Translation in NACA-TM-1256. 1908;56:1-37.

[3] Falkner VM, Skan SW. Some approximate solution of the boundary layer equations. Phil. Mag. 1931;12:865-896.

[4] Lighthill MJ. The response of laminar skin friction and heat transfer to fluctuations in the stream velocity. Proc. Roy. Soc. A. 1954;224:1-23.
.
[5] Career GF, Greenspan HP. The time-dependent magnetohydrodynamic flow past a flat plate. J. Fluid Mech. 1960;7:22-32.

[6] Das UN. A small unsteady perturbation on the steady hydromagnetic boundary layer flow past semi-infinite plate. Proc. Cambr. Phil. Soc. 1970;68:509-528.

[7] Meksyn D. Magneto-Hydrodynamic flow past a semi-infinite plate. ZAMP. 2005;17(3):397-403.

[8] Ingham DB. Flow past an impulsively started semi-infinite flat plate in the presence of a magnetic field. J Appl. Math. 1977;20(4):459-469.

[9] Animasaun IL, Prakash J, Vijayaragavan R, Sandeep N. Stagnation flow of nanofluid embedded with dust particles over an inclined stretching sheet with induced magnetic field and suction. Journal of Nanofluids. 2017;6:1–10.
DOI: 10.1166/jon.2017.1308

[10] Animasaun IL, Raju CSK, Sandeep N. Unequal diffusivities case of homogeneous–heterogeneous reactions within viscoelastic fluid flow in the presence of induced magnetic-field and nonlinear thermal radiation. Alexandria Engineering Journal. 2016;55:1595–1606.
DOI: 10.1016/j.aej.2016.01.018

[11] Hassanien IA. Flow and heat transfer on continuous stretching flat surface moving in a parallel free steam with variable properties. ZAAMZ. Angew Math. Mech. 1999;79:786-792.

[12] Mukhopadhyay S, Layek GC, Samad Sk. A. Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity. Int. J. Heat and Mass Transfer. 2005;48:4460-4466.

[13] Sarma U, Hazarika GC. Effects of variable viscosity and thermal conductivity on heat and mass transfer flow along a vertical plate in the presence of a magnetic field. Lat. Am. J. Phys. Educ. 2011;5(1):100-106.

[14] Salem AM. Variable viscosity and thermal conductivity effects on MHD flow and heat transfer in viscoelastic fluid over a stretching sheet. Phys. Lett. A. 2007;369(4):315–322.

[15] Seddeek MA, Salama FA. The effects of temperature dependent viscosity and thermal conductivity on unsteady MHD convective heat transfer past a semi-infinite vertical porous moving plate with variable suction. Comp. Mater. Sci. 2007;40(2):186–192.

[16] Odda SN, Farhan AM. Chebyshev finite difference method for the effects of variable viscosity and variable thermal conductivity on heat transfer to a micro-polar fluid from a non-isothermal stretching sheet with suction and blowing. Chaos Soliton. Fract. 2006;30(4):851–858.

[17] Pantokratoras A. Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity: A numerical reinvestigation. Int. J. of Heat and Mass Transfer. 2008;51:104-110.

[18] Lai FC, Kulacki FA. The effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous medium. Int. J. Heat Mass Transfer. 1990;33:1028–1031.

[19] Hazarika GC, Lahkar J. Effects of variable viscosity and thermal conductivity on free convective mass transfer flow past a vertical isothermal cone surface in presence of magnetic fiel. Mathematical Forum. 1997;xi:1-15.

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