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Discrete Dirichlet forms as traces of Feller's one-dimensional diffusions

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Abstract: Our primary purpose is to compute explicitly traces of the Dirichlet forms related to Feller's one-dimensional diffusions on countable sets via Fukushima's method. For discrete measures, the obtained trace form can be described as a Dirichlet form on the graph.

Keywords: Dirichlet forms; Trace operator; Feller's operator.

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1. Introduction

In this paper, we are interested in diffusion processes which have important applications in several domains such as biology, economy and finance. Thus from this research we draw our attention to discretize a Dirichlet form by computing its trace form through the scaling function θ as well as the speed measure μ associated to Feller's one-dimensional diffusions operators $\frac{d}{d\mu} \frac{d}{d\theta}$ on an open interval Ω .

In order to compute traces of Feller's Dirichlet forms on a countable set concerning a functional Tr , we shall apply the methods developed in [1,2] for the transient case. Hence in this situation, we have to determine the extended Dirichlet spaces \mathcal{D}_{ext} via Feller's classification of the boundaries. At the end of this paper, we will apply the obtained theoretical results to the Feller's Dirichlet form associated with the 1-order Bessel operator.

We arranged this paper as follows. In §2, we briefly review Feller's one-dimensional diffusions and basic notations to construct the Dirichlet form associated with Feller's operators. Then, in §3, we compute traces of Dirichlet form explicitly concerning discrete measures supported by countable sets F . Finally, examples in §4 such as 1-order Bessel processes on a weighted graph (\mathbb{N}, ω) illustrate the detailed results. We emphasize that a Bessel process on the half-line is also the radial component of the standard Brownian motion in the Euclidean space \mathbb{R}^n (we refer to [3] for more details).

2. Preliminaries

Let Ω be an open interval of the real line \mathbb{R} . We set $\mathcal{E}^{(\theta)}$ a Dirichlet form with domain $\mathcal{D}^{(\theta)}$ in $\mathcal{H} = L^2(\Omega, \mu)$ defined by

$$\mathcal{E}^{(\theta)}[f] := \int_{\Omega} \left(\frac{df}{d\theta}(x) \right)^2 d\theta(x).$$

Let a positive Radon measure ν with support $F \subset \Omega$ that charges no sets having zero capacity. We consider $\mathcal{H}_{aux} = L^2(F, \nu)$ and assume that trace operator

$$Tr : \text{dom}(Tr) \subseteq \mathcal{D}^{(\theta)} \cap L^2(F, \nu) \rightarrow L^2(F, \nu), f \mapsto f|_F,$$

is closed. We shall designate by \mathcal{E} the trace form of $\mathcal{E}^{(\theta)}$ w.r.t Tr (or w.r.t the measure ν) and by \mathcal{D} its domain.

2.1. Feller's Dirichlet forms

In this section we will introduce a Dirichlet form associated with the Feller's operator on some subsets of \mathbb{R} .

Let $\Omega = (0, \infty)$ and we consider a continuous strictly increasing function $\theta : \Omega \rightarrow \mathbb{R}$. Let

$$A_{loc}(\Omega) := \{f : \Omega \rightarrow \mathbb{R}, \text{ locally absolutely continuous on } \Omega\},$$

and

$$AC_\theta(\Omega) := \{f : \Omega \rightarrow \mathbb{R}, \text{ absolutely continuous w.r.t } d\theta \text{ on } \Omega\}.$$

Assume that $\theta \in AC_{loc}(\Omega)$ such that

$$\theta(x) = \int_c^x \gamma^{-1}(t) dt, \quad \forall x \in \Omega, c > 0,$$

where $\gamma > 0$ and $\gamma^{-1} = \frac{1}{\gamma} \in L^1_{loc}(\Omega)$. Obviously $\theta(dx) = \gamma^{-1}(x) dx$. Moreover, the scaling function θ can be also regarded as a scaling measure on Borel subsets J of \mathbb{R} given by

$$\theta(J) := \int_J \gamma^{-1}(x) dx.$$

Giving a speed measure μ with full support Ω as

$$\mu(dx) = \gamma(x) dx,$$

with $\gamma > 0$ and $\gamma \in L^1_{loc}(\Omega)$. We denote by

$$\mathcal{D}^{(\theta)} := \left\{ f : \Omega \rightarrow \mathbb{R} : f \in AC_\theta(\Omega), \int_\Omega \left(\frac{df}{d\theta}(x) \right)^2 d\theta < \infty \right\},$$

where $\frac{df}{d\theta}$ is the Radon-Nikodym derivative of df w.r.t $d\theta$.

We define a Dirichlet form \mathcal{E} with a densely defined domain \mathcal{D} in $L^2(\Omega, \mu)$ by

$$\mathcal{D} := \mathcal{D}^{(\theta)} \cap L^2(\Omega, \mu), \quad \mathcal{E}[f] := \int_\Omega \left(\frac{df}{d\theta}(x) \right)^2 d\theta \quad \text{for all } f \in \mathcal{D}. \tag{1}$$

Lemma 1. *Each function from \mathcal{D} is continuous and a.e. differentiable on Ω .*

Proof. As each function $f \in \mathcal{D}$ is absolutely continuous w.r.t $d\theta$, it is a composition of continuous functions, hence continuous. Since θ is strictly increasing then by Lebesgue’s theorem it is a.e. differentiable. Besides every locally absolutely continuous function is a.e. differentiable. Hence f is a.e. differentiable. \square

Using Lemma 1 together with the fact that $\frac{1}{\gamma} \in L^1_{loc}(\Omega)$, we get

$$\frac{df}{d\theta}(x) = \frac{df}{dx} \cdot \frac{dx}{d\theta}(x) = \frac{df}{dx} \cdot \gamma(x).$$

Hence

$$\mathcal{D}^{(\theta)} = \left\{ f : \Omega \rightarrow \mathbb{R} : f \in AC_\theta(\Omega), \int_\Omega \left(\frac{df}{dx}(x) \right)^2 \gamma(x) dx < \infty \right\}, \tag{2}$$

and

$$\mathcal{D} := \mathcal{D}^{(\theta)} \cap L^2(\Omega, \mu), \quad \mathcal{E}[f] := \int_\Omega \left(\frac{df}{dx}(x) \right)^2 \gamma(x) dx \quad \text{for all } f \in \mathcal{D}. \tag{3}$$

We claim that \mathcal{E} is a regular strongly local Dirichlet form in $L^2(\Omega, \mu)$ [4]. According to the first Kato’s representation Theorem [5] we can associate to \mathcal{E} a positive self-adjoint operator L which is defined by (we refer to [6, §2] and [1, Chapter 1])

$$\begin{aligned} \text{dom}(L) &= \left\{ f \in \mathcal{D} : \frac{df}{dx} \in AC_{loc}(\Omega), \lim_{x \rightarrow 0^+} \frac{df}{d\theta}(x) = 0, \Delta_{\mu,\theta} f = -\frac{d}{d\mu} \left(\frac{df}{d\theta} \right) \in L^2(\Omega, \mu) \right\} \\ Lf &= \Delta_{\mu,\theta} f \quad \text{for all } f \in \text{dom}(L). \end{aligned}$$

Correspondingly the Feller’s canonical operator $\Delta_{\mu,\theta}$ is defined as (see [4, p. 63-64])

$$\Delta_{\mu,\theta} f := -\frac{d}{d\mu} \left(\frac{df}{d\theta} \right) = -\frac{1}{\gamma} \frac{d}{dx} \cdot \left(\gamma \frac{df}{dx} \right) = -\left(\frac{d^2 f}{dx^2} + \frac{\gamma'}{\gamma} \frac{df}{dx} \right).$$

2.2. Feller's classification of the boundaries

We let

$$\lambda_1 = \int_0^c \left(\int_x^c \gamma(y) dy \right) \frac{1}{\gamma(x)} dx \quad \text{and} \quad \lambda_2 = \int_0^c \left(\int_c^x \frac{1}{\gamma(y)} dy \right) \gamma(x) dx, \quad (4)$$

for some $c > 0$.

The left boundary 0 can be classified as follows: (see [7, p. 151-152])

- (a) regular if $\lambda_1 < \infty$, $\lambda_2 < \infty$,
- (b) exit if $\lambda_1 < \infty$, $\lambda_2 = \infty$,
- (c) entrance if $\lambda_1 = \infty$, $\lambda_2 < \infty$,
- (d) natural if $\lambda_1 = \infty$, $\lambda_2 = \infty$.

Analogously to the right boundary ∞ .

Definition 1 ([4]). (a) We say that the boundary 0 (resp. ∞) is approachable whenever

$$\theta(0) > -\infty \quad (\text{resp. } \theta(\infty) < \infty).$$

- (b) We say that the boundary 0 (resp. ∞) is regular if

$$\text{it is approachable and } \int_0^c \gamma(y) dy < \infty, \quad (\text{resp. } \int_c^\infty \gamma(y) dy < \infty), \quad \forall c > 0.$$

We designate by \mathcal{D}_{ext} the extended Dirichlet space of \mathcal{E} which is introduced in [4].

Remark 1 ([4]). If a Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(\Omega, \mu)$ is transient, then its extended Dirichlet space \mathcal{D}_{ext} is complete w.r.t the metric \mathcal{E} .

Henceforth, making use the theorem below to induce the extended Dirichlet space \mathcal{D}_{ext} .

Theorem 1. Assume that boundary 0 (resp. ∞) is approachable but non-regular. Let

$$\mathcal{D}_0^{(\theta)} := \{f \in \mathcal{D}^{(\theta)} : f(0) = 0\} \quad (\text{resp. } \mathcal{D}_\infty^{(\theta)} := \{f \in \mathcal{D}^{(\theta)} : f(\infty) = 0\}) \quad (5)$$

$$(\mathcal{E}, \mathcal{D}) = (\mathcal{E}^{(\theta)}, \mathcal{D}_0^{(\theta)} \cap L^2(\Omega, \mu)) \quad (6)$$

is a transient Dirichlet form on $L^2(\Omega, \mu)$. Therefore

$$\mathcal{D} \subset \mathcal{D}_{ext} = \mathcal{D}_0^{(\theta)}, \quad \mathcal{E} = \mathcal{E}^{(\theta)}. \quad (7)$$

Proof. See [8, Theorem 3.2]. \square

By virtue of [4, Theorem 2.2.11] \mathcal{E} is transient if and only if either 0 or ∞ is approachable and non-regular. Henceforth we assume that either 0 or ∞ is approachable but non-regular. Thus \mathcal{E} is transient. The Feller's test of non-explosion processes ([4, p. 126]) leads to the conservativeness property of \mathcal{E} whenever

$$\int_0^c \left(\int_x^c \gamma(y) dy \right) \frac{1}{\gamma(x)} dx = \int_c^\infty \left(\int_c^x \gamma(y) dy \right) \frac{1}{\gamma(x)} dx = \infty, \quad \forall c > 0. \quad (8)$$

We remark that endpoints 0 and ∞ are non-exit.

Remark 2. If 0 (resp. ∞) is approachable, hence for each function f from $\mathcal{D}^{(\theta)}$ we get $f(0) = \lim_{x \downarrow 0, x \in \Omega} f(x) < \infty$, (resp. $f(\infty) = \lim_{x \uparrow \infty, x \in \Omega} f(x) < \infty$) and $f \in \mathcal{C}(\overline{\Omega})$, (resp. $f \in \mathcal{C}(\Omega)$) [4].

3. Traces of one-dimensional diffusions on countable sets

Let $F = \{p_k \in \Omega, k \in \mathbb{N}\}$ be a countable set. Let $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence of in the right half axis $(0, \infty)$. We define a discrete measure ν on F by

$$\nu = \sum_{k \in \mathbb{N}} \alpha_k \delta_{p_k},$$

where δ_{p_k} denote the Dirac measure centred on some fixed point p_k in F .

Let $\mathcal{H}_{aux} = \ell^2(F, \mu)$ be a real Hilbert space equipped with a scalar product given by

$$(f, g)_\nu := \sum_{k \in \mathbb{N}} \alpha_k f(p_k) g(p_k) \quad \text{and} \quad \|f\|_\nu := \sqrt{(f, f)_\nu}.$$

On the other hand, $\check{\mathcal{E}}$ is the trace of \mathcal{E} on F w.r.t ν (see [1,9,10]). As by assumptions \mathcal{E} is transient, hence we can adopt the technique developed in [2] to evaluate $\check{\mathcal{E}}$.

We shall now discuss the following case:

0 is approachable and non-regular.

According to Theorem 1 the extended domain \mathcal{D}_{ext} is given by

$$\mathcal{D}_{ext} = \left\{ f: \Omega \rightarrow \mathbb{R} : f \in AC_\theta(\Omega), \int_\Omega \left(\frac{df}{dx}(x) \right)^2 \gamma(x) dx < \infty \text{ s.t. } f(0) = 0 \right\}.$$

The trace operator Tr defined from $\text{dom}(Tr)$ to $\ell^2(F, \nu)$ by

$$\begin{aligned} \text{dom}(Tr) &:= \left\{ f \in \mathcal{D}_{ext} : \sum_{k \in \mathbb{N}} \alpha_k f^2(p_k) < \infty \right\}, \\ Tr f &:= f|_F \quad \text{for all } f \in \text{dom}(Tr). \end{aligned}$$

Furthermore, the linear operator Tr is bounded on $(\mathcal{D}_{ext}, \mathcal{E})$ if and only if (α_k) is bounded. Regularity of \mathcal{E} implies density of $\text{dom}(Tr)$ in $(\mathcal{D}_{ext}, \mathcal{E})$ as well as density of $\text{ran}(Tr)$ in $\ell^2(F, \nu)$. Obviously

$$\ker(Tr) = \left\{ f \in \text{dom}(Tr) : f(p_k) = 0 \quad \text{for all } k \in \mathbb{N} \right\}.$$

According to [1, Chapter 1] we can compute the trace form $\check{\mathcal{E}}$ following method introduced by Fukushima. Thereby we denote by Π the \mathcal{E} -orthogonal projection in \mathcal{D}_{ext} onto $(\ker(Tr))^\perp$. Then

$$\text{dom } \check{\mathcal{E}} = \text{ran}(Tr) \quad \text{and} \quad \check{\mathcal{E}}[Tr f] = \mathcal{E}[\Pi f], \quad \text{for all } f \in \text{dom}(Tr).$$

Certainly

$$(\ker(Tr))^\perp = \left\{ f \in \mathcal{D}_{ext} : \mathcal{E}(f, g) = 0, \quad \text{for all } g \in \ker(Tr) \right\}.$$

We are now in position to compute \mathcal{E} -orthogonal projection Π .

Theorem 2. *Let $f \in \mathcal{D}_{ext}$. Then Πf a solution from \mathcal{D}_{ext} of the Sturm-Liouville problem*

$$\begin{aligned} -(\Pi f)'' - \frac{\gamma'}{\gamma}(\Pi f)' &= 0 \quad \text{in } \Omega \setminus F, \\ \Pi f &= f \quad \text{on } F. \end{aligned} \tag{9}$$

Proof. Assume that $t = \theta^{-1}$ is absolutely continuous and $t' \in L^2_{loc}(\Omega)$. Hence, making use [11, Lemma 2.1] we get

$$\mathcal{C}_c^\infty(\Omega \setminus F) \subset \mathcal{D}.$$

Let $f \in \mathcal{D}_{ext}$. Given that Π the \mathcal{E} -orthogonal projection from \mathcal{D}_{ext} onto $(\ker(Tr))^\perp$, then

$$\mathcal{E}(\Pi f, g) = 0, \quad \text{for all } g \in \mathcal{C}_c^\infty(\Omega \setminus F),$$

which is equivalent to

$$\left\langle \frac{1}{\gamma}(\Pi f)', g' \right\rangle_{\mathcal{C}_c^\infty(\Omega \setminus F)', \mathcal{C}_c^\infty(\Omega \setminus F)} = 0,$$

with $\langle \cdot, \cdot \rangle_{C_c^\infty(\Omega \setminus F)', C_c^\infty(\Omega \setminus F)}$ is the dual bracket between $C_c^\infty(\Omega \setminus F)$ and its dual $C_c^\infty(\Omega \setminus F)'$. Accordingly, we obtain

$$-\langle (\frac{1}{\gamma}(\Pi f)')', g \rangle_{C_c^\infty(\Omega \setminus F)', C_c^\infty(\Omega \setminus F)} = 0,$$

which leads to

$$-(\Pi f)'' - \frac{\gamma'}{\gamma}(\Pi f)' = 0 \text{ in } C_c^\infty(\Omega \setminus F)'. \tag{10}$$

On the other hand, the closedness of the operator Tr yields the closedness of $Ker(Tr)$ as well. Hence $f - \Pi f \in Ker(Tr)$ and $Tr f = Tr \Pi f$. Thus $f = \Pi f$ on F .

It is easy to proof the converse so we omit.

□

We are in position now to compute explicitly the \mathcal{E} -orthogonal projection Πf solution of the boundary value problem (9). Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 2. *Let $f \in \mathcal{D}_{ext}$. Then*

$$\Pi f(x) = f(p_k) + \Gamma_k(x) \cdot \frac{f(p_{k+1}) - f(p_k)}{\theta([p_k, p_{k+1}])} \tag{11}$$

where $\Gamma_k(x) = \int_{p_k}^x \gamma^{-1}(\tau) d\tau$ for all $x \in [p_k, p_{k+1}]$ and $k \in \mathbb{N}_0$.

Proof. On the light of the Theorem 2, it suffices to show that the function given by (11) solves the following ODE

$$-(\Pi f)''(x) - \frac{\gamma'(x)}{\gamma(x)}(\Pi f)'(x) = 0, \text{ in } (0, \infty). \tag{12}$$

Indeed the solution of the homogeneous differential equation (12) is given by

$$\Pi f(x) = M_k + N_k \int_{p_k}^x \gamma^{-1}(\tau) d\tau, \text{ in } [p_k, p_{k+1}], \text{ for all } k \in \mathbb{N}, \tag{13}$$

where $M_k, N_k \in \mathbb{R}$. Hence, we obtain

$$\begin{aligned} \Pi f(p_k) &= M_k = f(p_k), \\ \Pi f(p_{k+1}) &= M_k + N_k \int_{p_k}^{p_{k+1}} \gamma^{-1}(\tau) d\tau = f(p_{k+1}). \end{aligned}$$

This leads to achieve

$$\begin{aligned} \Pi f(x) &= f(p_k) + \left(\int_{p_k}^x \gamma^{-1}(\tau) d\tau \right) \cdot \frac{f(p_{k+1}) - f(p_k)}{\int_{p_k}^{p_{k+1}} \gamma^{-1}(\tau) d\tau} \\ &= f(p_k) + \Gamma_k(x) \cdot \frac{f(p_{k+1}) - f(p_k)}{\theta([p_k, p_{k+1}])}, \text{ for all } x \in [p_k, p_{k+1}], k \in \mathbb{N}_0. \end{aligned}$$

Since $\Pi f \in \mathcal{D}_{ext}$ we have $\Pi f(0) = 0$.

□

Theorem 3. *For each function $f \in \text{dom}(Tr)$. It holds $\text{dom}(\check{\mathcal{E}}) = \text{ran}(Tr)$ and*

$$\begin{aligned} \check{\mathcal{E}}[Tr f] &= \sum_{k \in \mathbb{N}_0} \left[\frac{d\Pi f}{dx}(p_{k+1}^-) f(p_{k+1}) \gamma(p_{k+1}) - \frac{d\Pi f}{dx}(p_k^+) f(p_k) \gamma(p_k) \right] \\ &= \sum_{k \in \mathbb{N}_0} \frac{1}{\theta([p_k, p_{k+1}])} (f(p_{k+1}) - f(p_k))^2, \end{aligned} \tag{14}$$

where $\frac{d\Pi f}{dx}(p_k^+)$ and $\frac{d\Pi f}{dx}(p_{k+1}^-)$ are the right derivative at p_k and the left derivative at p_{k+1} respectively.

Proof. Let $f \in \text{dom}(Tr)$ and $p_0 = 0$. An integration by parts leads to

$$\begin{aligned} \check{\mathcal{E}}[Tr f] &= \mathcal{E}[\Pi f] = \int_{\Omega} \left(\frac{d\Pi f}{dx}(x)\right)^2 \gamma(x) dx \\ &= \sum_{k \in \mathbb{N}_0} \int_{p_k}^{p_{k+1}} \left(-\frac{d^2\Pi f}{dx^2}(x) - \frac{\gamma'(x)}{\gamma(x)} \frac{d\Pi f}{dx}(x)\right) (\Pi f)(x) \gamma(x) dx + \sum_{k \in \mathbb{N}_0} \frac{d\Pi f}{dx}(x) (\Pi f)(x) \gamma(x) \Big|_{p_k}^{p_{k+1}} \\ &= \sum_{k \in \mathbb{N}_0} \left[\frac{d\Pi f}{dx}(p_{k+1}^-) f(p_{k+1}) \gamma(p_{k+1}) - \frac{d\Pi f}{dx}(p_k^+) f(p_k) \gamma(p_k) \right]. \end{aligned} \tag{15}$$

On the other hand we compute the derivative $\frac{d\Pi f}{dx}$ for all $x \in [p_k, p_{k+1}]$ and $k \in \mathbb{N}$,

$$\frac{d\Pi f}{dx}(x) = \frac{d\Gamma_k(x)}{dx} \cdot \frac{f(p_{k+1}) - f(p_k)}{\theta([p_k, p_{k+1}])} = \gamma^{-1}(x) \cdot \frac{f(p_{k+1}) - f(p_k)}{\theta([p_k, p_{k+1}])}, \text{ in } [p_k, p_{k+1}].$$

Putting all together we obtain

$$\begin{aligned} \check{\mathcal{E}}[Tr f] &= \sum_{k \in \mathbb{N}_0} \frac{1}{\int_{p_k}^{p_{k+1}} \gamma^{-1}(\tau) d\tau} \cdot (f(p_{k+1}) - f(p_k))^2 \\ &= \sum_{k \in \mathbb{N}_0} \frac{1}{\theta([p_k, p_{k+1}])} (f(p_{k+1}) - f(p_k))^2, \text{ for all } f \in \text{dom}(Tr), \end{aligned}$$

which finishes the proof. \square

3.1. Feller’s Dirichlet forms on graphs

At this stage, the trace form $\check{\mathcal{E}}$ can be rewritten as

$$\check{\mathcal{E}}[Tr f] = \sum_{p_k \in F} \sum_{p_j \sim p_k} \omega(p_k, p_j) (f(p_k) - f(p_j))^2, \text{ for each } p_j \in F,$$

where

$$\omega(p_k, p_j) = \frac{1}{2\theta([p_k, p_j])} > 0 \text{ if } p_k \sim p_j, \text{ and } \omega(p_k, p_j) = 0 \text{ otherwise.}$$

Then

$$\check{\mathcal{E}}[Tr f] = \sum_{k \in \mathbb{N}} \omega(p_k, p_{k+1}) (f(p_{k+1}) - f(p_k))^2, \text{ for all } f \in \text{dom}(Tr).$$

We shall now adapt the geometric condition (A) of [12]. Moreover, $\check{L}(\mathcal{C}_c(F)) \subseteq \mathcal{C}_c(F)$ where

$$\check{L}f(p_k) = \frac{1}{\alpha_k} \sum_{j \sim k} \omega(p_k, p_j) (f(p_k) - f(p_j)), \text{ for each } k \in \mathbb{N}. \tag{16}$$

By virtue of Theorem 6 from [12] we achieve \check{L} as

$$\begin{aligned} \text{dom}(\check{L}) &:= \left\{ f \in \ell^2(F, \nu) : \check{L}f \in \ell^2(F, \nu) \right\} \\ \check{L}f &= \check{L}f. \end{aligned}$$

It holds for all $k \in \mathbb{N}$ and $f \in \text{dom}(\check{L})$

$$\check{L}f(p_k) = -\frac{1}{\alpha_k} \left(\frac{f(p_{k+1})}{\theta([p_k, p_{k+1}])} - \frac{\theta([p_{k-1}, p_{k+1}])f(p_k)}{\theta([p_k, p_{k+1}])\theta([p_{k-1}, p_k])} + \frac{f(p_{k-1})}{\theta([p_{k-1}, p_k])} \right). \tag{17}$$

Remark 3. Assume that $\alpha_k = 1$ for all $k \in \mathbb{N}$. Let us emphasize that the latter formulae can be viewed as a discrete Jacobi operator as follows,

$$J\phi_k := A_k\phi_{k+1} + B_k\phi_k + A_{k-1}\phi_{k-1}, \forall k \in \mathbb{N}, \tag{18}$$

and

$$\phi_0 = 0. \tag{19}$$

4. Examples

4.1. Traces of one-dimension diffusion operators on \mathbb{N}

We consider the state space $I = [0, \infty)$. Let θ be a canonical scaling function (i.e. $\theta(x) = x$) and μ be a canonical speed measure such that $supp[\mu] = I$. Let us consider $\mathcal{H} := L^2((0, \infty), dx)$, $\mathcal{D} = W^{1,2}(0, \infty)$ and define \mathcal{E} in \mathcal{H} by

$$\mathcal{D} := W^{1,2}(0, \infty), \quad \mathcal{E}[f] := \int_0^\infty \left(\frac{df}{dx}(x)\right)^2 dx.$$

It is known that \mathcal{E} is associated with the Laplacian on $(0, \infty)$ with Neumann condition on $(0, \infty)$ that is $\frac{df}{dx}(0^+) = 0$. Furthermore from [4, Theorem 2.2.11, p.68] \mathcal{E} is recurrent and hence conservative.

Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $(0, \infty)$ and $\nu := \sum_{n \in \mathbb{N}} \alpha_n \delta_n$. Choose $\mathcal{H}_{aux} = L^2(F, \nu) = \ell^2(\mathbb{N}, \nu)$. By Sobolev’s embedding theorem, every $f \in W^{1,2}(0, \infty)$ has a unique continuous representative \tilde{f} . We shall assume that every element in $W^{1,2}(0, \infty)$ is continuous. The trace operator Tr is defined from \mathcal{D} to $\ell^2(\mathbb{N}, \nu)$ by

$$\text{dom}(Tr) := \{f \in W^{1,2}(0, \infty) : \sum_{n \in \mathbb{N}} \alpha_n f_n^2 < \infty\}, \quad Tr f := f|_{\mathbb{N}} \quad \text{for all } f \in \text{dom}(Tr).$$

A routine computation leads to

$$\check{\mathcal{D}} := \text{dom } \check{\mathcal{E}} = \{\psi = (\psi_n) \in \ell^2(\mathbb{N}, \nu), \sum_{n \in \mathbb{N}} (\psi_{n+1} - \psi_n)^2 < \infty\},$$

and

$$\begin{aligned} \check{\mathcal{E}}[\psi] &= \sum_{n \in \mathbb{N}} \frac{1}{\theta([n, n+1])} (\psi_{n+1} - \psi_n)^2 \\ &= \sum_{n \in \mathbb{N}} \frac{1}{((n+1) - n)} (\psi_{n+1} - \psi_n)^2 \\ &= \sum_{n \in \mathbb{N}} (\psi_{n+1} - \psi_n)^2. \end{aligned}$$

Owing to these considerations, $\check{\mathcal{E}}$ has the following expression

$$\check{\mathcal{E}}[\psi] = \sum_{n \in \mathbb{N}} \sum_{m \sim n} \omega(n, m) (\psi_n - \psi_m)^2, \quad \text{for each } m \in \mathbb{N}, \tag{20}$$

where $\omega(n, m) = 1/2$ if $|n - m| = 1$ and $\omega(n, m) = 0$ for $|n - m| > 1$.

From [12, Theorem 6] the discrete Laplacian operator associated with $\check{\mathcal{E}}$ is given by

$$(\check{L}\psi)_n = -\psi_{n+1} + 2\psi_n - \psi_{n-1}, \quad \text{for all } n \in \mathbb{N},$$

and acting on

$$\text{dom}(\check{L}) := \{\psi = (\psi_n) \in \ell^2(\mathbb{N}, \nu) : \check{L}\psi \in \ell^2(\mathbb{N}, \nu)\}.$$

4.2. Traces of the Feller’s Dirichlet forms related to 1-order Bessel’s process on \mathbb{N}

Let us consider a speed measure μ defined on $\Omega = (0, \infty)$ by

$$\mu(dx) = 2x^3 dx.$$

We define the scaling function θ on Ω by

$$\theta(dx) = \frac{1}{x^3} dx.$$

We are concerned with the Feller's Dirichlet form \mathcal{E} with domain $\mathcal{D} \subset \mathcal{H} = L^2(\Omega, \mu)$ defined by

$$\mathcal{D} := \mathcal{D}^{(\theta)} \cap L^2(\Omega, \mu), \quad \mathcal{E}[f] := \int_{\Omega} \left(\frac{df}{dx}(x)\right)^2 x^3 dx \quad \text{for all } f \in \mathcal{D},$$

where

$$\mathcal{D}^{(\theta)} := \left\{ f: \Omega \rightarrow \mathbb{R} : u \in AC_{\theta}(\Omega), \int_{\Omega} \left(\frac{df}{dx}(x)\right)^2 x^3 dx < \infty \right\}.$$

It is easy to check that boundary 0 is non-approachable whereas ∞ is approachable and non-regular. Accordingly, we can determine the extended domain \mathcal{D}_{ext} as

$$\mathcal{D}_{ext} = \{f \in \mathcal{D}^{(\theta)} : f(\infty) = 0\}. \quad (21)$$

Let us consider the 1-order Bessel operator defined by

$$\Delta_{\mu, \theta} := -\left(\frac{d^2}{dx^2} + \frac{3}{2x} \frac{d}{dx}\right).$$

Under those circumstances, we give the self-adjoint operator L associated with \mathcal{E} via Kato's representation theorem by

$$\begin{aligned} \text{dom}(L) &= \left\{ f \in \mathcal{D} : \frac{df}{dx} \in AC_{loc}(\Omega), \lim_{x \downarrow 0^+} x^3 \frac{df}{dx}(x) = 0, \Delta_{\mu, \theta} f \in L^2(\Omega, \mu) \right\} \\ Lf &= \Delta_{\mu, \theta} f \quad \text{for all } f \in \text{dom}(L). \end{aligned}$$

Hence, L is nothing else but the generator of the 1-order Bessel process on Ω .

Accordingly we fix a discrete measure

$$\nu = \sum_{n \in \mathbb{N}} \alpha_n \delta_n,$$

which is supported by $F = \mathbb{N}$.

In order to compute $\check{\mathcal{E}}$ on the set \mathbb{N} in $\mathcal{H}_{aux} = \ell^2(\mathbb{N}, \nu)$ we proceed to apply Theorem 3 to get the following formulae

$$\text{dom } \check{\mathcal{E}} = \text{ran}(Tr), \quad (22)$$

$$\begin{aligned} \check{\mathcal{E}}[Tr f] &= \sum_{n \in \mathbb{N}} \frac{1}{\theta([n, n+1])} (f_{n+1} - f_n)^2 \\ &= \sum_{n \in \mathbb{N}} \frac{1}{\int_n^{n+1} x^{-3} dx} (f_{n+1} - f_n)^2 \\ &= \sum_{n \in \mathbb{N}} 2 \frac{(n(n+1))^2}{2n+1} (f_{n+1} - f_n)^2, \quad \text{for all } f \in \text{dom}(Tr). \end{aligned} \quad (23)$$

Let us rewrite $\check{\mathcal{E}}$ in the following way

$$\check{\mathcal{E}}[Tr f] = \sum_{n \in \mathbb{N}} \sum_{n \sim m} \omega(n, m) (f_n - f_m)^2, \quad \text{for each } m \in \mathbb{N}, \quad (24)$$

where

$$\omega(n, m) = \frac{(nm)^2}{n+m} > 0 \quad \text{if } |n-m| = 1, \quad \text{and } \omega(n, m) = 0 \quad \text{for } |n-m| > 1.$$

Consequently, by [12, Theorem 6] we can describe $\check{\mathcal{E}}$ as follows:

$$\check{\mathcal{D}} := \text{dom } \check{\mathcal{E}} = \left\{ f = (f_n) \in \ell^2(\mathbb{N}, \nu), \sum_{n \in \mathbb{N}} 2 \frac{(n(n+1))^2}{2n+1} (f_{n+1} - f_n)^2 < \infty \right\}, \quad (25)$$

$$\check{\mathcal{E}}[f] = \sum_{n \in \mathbb{N}} 2 \frac{(n(n+1))^2}{2n+1} (f_{n+1} - f_n)^2. \quad (26)$$

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