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Schur Geometric Convexity for Ratio of Difference of Means

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Authors' contributions

This work was carried out in collaboration between all authors. Author VL Proposed the concept and carried out with BNK, wrote the first draft of the manuscript. Authors KMN and SP managed the literature searches, analyses of the study. All authors read and approved the final manuscript.

Original Research Article

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ABSTRACT

In this paper, we study the Schur geometric convexity (concavity) of the ratio of difference of means. Also established some inter related mean inequalities related to the ratio of difference of means.

Keywords: Schur convexity; schur harmonic convexity; ratio of difference of means; inequality.

1. INTRODUCTION

The well-known means in literature such as arithmetic mean, geometric mean harmonic mean, Heron means and contra harmonic mean are presented by pappus of Alexandria [1,2]. In Pythagorean School on the basis of proportion these means are defined as follows:

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For a, b are positive real numbers,

$$A(\mathrm{a}, \mathrm{b}) = \frac{a+b}{2} \tag{1}$$

$$G(a, b) = \sqrt{ab} \tag{2}$$

$$H(a, b) = \frac{2ab}{a+b} \tag{3}$$

And

$$C(a, b) = \frac{a^2+b^2}{a+b} \tag{4}$$

are respectively called arithmetic mean, geometric mean, harmonic mean and contra harmonic mean.

For positive real numbers a and b the Heron mean is defined as:

$$H_e(a, b) = \frac{a+\sqrt{ab}+b}{3} \tag{5}$$

Let $a = t, b = 1$ in equations (1)-(5). Then

$$A(\mathrm{t}, 1) = \frac{t+1}{2} \tag{6}$$

$$G(t, 1) = \sqrt{t} \tag{7}$$

$$H(t, 1) = \frac{2t}{t+1} \tag{8}$$

$$C(t, 1) = \frac{t^2+1}{t+1} \tag{9}$$

And

$$H_e(t, 1) = \frac{t+\sqrt{t}+1}{3} \tag{10}$$

Various researchers have studied several homogeneous functions and obtained identities involving means and established remarkable mean inequalities [3 -8].

In [9], Jamal Rooin and Mehdi Hassni, introduced the homogeneous functions
f(x) and g(x), where

$$f(x) = \frac{a^x-b^x}{c^x-d^x} \quad \text{and} \quad g(x) = ln\frac{a^x-b^x}{c^x-d^x} \;\text{ for }\; x \in (-\infty, \infty) \tag{11}$$

and $\qquad a > b \geq c > d > 0.$

Further, authors established some convexity results and refinements to Ky-Fan-type inequalities. Motivated by the attempt to introduce the ratio of difference of means and to establish some inequalities involving them. And we have studied the convexity(concavity) of the following ratio of difference of means see [10].

$$M_{CAGH}(\mathrm{a}, \mathrm{b}) = \frac{C(a,b)-A(a,b)}{G(a,b)-H(a,b)} \tag{12}$$

$$M_{AH_eGH}(\mathrm{a,b}) = \frac{A(a,b)-H_e(a,b)}{G(a,b)-H(a,b)} \tag{13}$$

$$M_{CH_eGH}(\mathrm{a,b}) = \frac{C(a,b)-H_e(a,b)}{G(a,b)-H(a,b)} \tag{14}$$

And

$$M_{CAH_eG}(\mathrm{a,b}) = \frac{C(a,b)-A(a,b)}{H_e(a,b)-G(a,b)} \tag{15}$$

In this paper, we study the Schur geometric convexity of ratio of difference of means $M_{CAGH}(\mathrm{a,b})$, $M_{AH_eGH}(\mathrm{a,b})$, $M_{CH_eGH}(\mathrm{a,b})$, $M_{CAH_eG}(\mathrm{a,b})$. and some applications of these ratio of difference of means.

2. PRELIMINARY RESULTS

In 1923, the Schur Convex function was introduced by I Schur, and proved many important applications to analytic inequalities. In 2003, X. M. Zhang propose the concept of Schur-geometrically convex function which is an extension of Schur-convexity function. In recent years, the Schur convexity, Schur geometrically convexity and Schur harmonic convexity have attracted the attention of a considerable number of mathematicians ([11],- [21]). For convenience of readers, we recall some definitions as follows:

Definition 1. [3,7] Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in R^n$.

1. Let x is said to be majorized by y (in symbol $x \prec y$) $\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i$ for $k = 1,2,3,\dots,n$ and $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i$ where $x_{[1]} \ge, \dots, \ge x_{[n]}$ and $y_{[1]} \ge, \dots, \ge y_{[n]}$ are rearrangement of $x\ and\ y$ in descending order.
2. $\Omega \subseteq R^n$. The function $\varphi: \Omega \to R$ is said to be schur convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \le \varphi(y)$. φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex.

Definition 2. [22] Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in R_+^n$. $\Omega \subseteq R^n$ is called geometrically convex set if $(x_1^{\alpha} y_1^{\beta}, \dots, x_n^{\alpha} y_n^{\beta}) \in R^n$ for all $x\ and\ y \in \Omega$ where $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

Let $\Omega \subseteq R_+^n$. The function $\varphi: \Omega \to R_+$ is said to be schur geometrically convex function on Ω if $(ln x_1, \dots, ln x_n) \prec (ln y_1, \dots, ln y_n)$ on Ω implies $\varphi(x) \le \varphi(y)$. Let φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex.

Definition 3. ([3], [7]) The set $\Omega \subseteq R^n$ is called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

The function $\varphi: \Omega \to R$ is said to be symmetric if every permutation matrix P,
$\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Lemma 1. ([3], [7]) Let $\Omega \subseteq R^n$. $\varphi: \Omega \to R$ is symmetric and convex function. Then φ is Schur convex on Ω.

Remark 1. The ratios of difference means given (12)-(15) are symmetric.

Lemma 2. [11] For $a > b \geq c > d > 0$, the function $f(x) = \frac{a^x - b^x}{c^x - d^x}$ where $x \in (-\infty, \infty)$ is

(i) Convex, if $ad - bc > 0$
(ii) Concave if $ad - bc < 0$
(iii) Equality holds if $ad - bc = 0$.

Lemma 3. [22] Let $\Omega \subseteq R^n$ be symmetric with non empty interior geometrically convex set and let $\varphi: \Omega \to R_+$ be continuous on Ω and differentiable on Ω^0. If φ is symmetric on Ω and

$$S = (lnx_1 - lnx_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0). \tag{16}$$

Holds for any on $x = (x_1, x_2, \dots, x_n) \in \Omega^0$ then φ is a Schur geometrically convex (Schur geometrically concave) function.

Lemma 4. [17] Let $a \leq b,\ u(t) = ta + (1-t)b,\ v(t) = tb + (1-t)a,\ \frac{1}{2} \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$. Then

$$\frac{(a+b)}{2} \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)). \tag{17}$$

Theorem 1. For $a, b > 0$, then the ratio of difference of meansM_{AH_eGH} is schur geometrically concave.

Proof. Recall equation (13)

$$M_{AH_eGH}(\mathrm{a}, \mathrm{b}) = \frac{A(a,b) - H_e(a,b)}{G(a,b) - H(a,b)}.$$

From Lemma 2, if $AH - GH_e < 0$ then M_{AH_eGH} is concave.
Consider,

$$f(a,b) = AH - GH_e = \frac{a+b}{2}\frac{2ab}{a+b} - \frac{a + \sqrt{ab} + b}{3}\sqrt{ab}$$

$$f(a,b) = \frac{2ab - (a+b)\sqrt{ab}}{3}$$

By finding the partial derivatives of $f(a,b)$ and with simple manipulation gives

$$a\frac{\partial f}{\partial a} = \frac{1}{3}\left[2ab - \frac{ab^2 + 3a^2b}{2\sqrt{ab}}\right]$$

And

$$b\frac{\partial f}{\partial b} = \frac{1}{3}\left[2ab - \frac{a^2b + 3ab^2}{2\sqrt{ab}}\right]$$

From Lemma 3,

$$(lna - lnb)\left(a\frac{\partial f}{\partial a} - b\frac{\partial f}{\partial b}\right) = \frac{(lna - lnb)}{3}(a-b)\sqrt{ab} \leq 0$$

Holds for $a \geq b$.

This completes the proof of theorem 1.

Remark1. For $a = t, b = 1$, in the equation (16) gives

$$S = \frac{\log t}{3}\left(t^{\frac{1}{2}} - t^{\frac{3}{2}}\right) = f(t)$$

$$f^{'}(t) = \frac{1}{6}(\log t + 2)\left(t^{\frac{1}{2}} - t^{\frac{3}{2}}\right)$$

$$f^{''}(t) = -\frac{1}{6}t^{-\frac{3}{2}}\left(\frac{1}{2}\log(1+3t^2) + 6t^2\right)$$

Theorem 2. For $a, b > 0$, the ratio of difference of means M_{CH_eGH} is schur geometrically concave.

Proof. Recall equation (12)

$$M_{CH_eGH}(\mathrm{a,b}) = \frac{C(a,b) - H_e(a,b)}{G(a,b) - H(a,b)}.$$

From Lemma 2, if $CH - AG < 0$ then M_{AH_eGH} is concave.
Consider,

$$f(a,b) = CH - GH_e = \frac{a^2+b^2}{a+b}\,\frac{2ab}{a+b} - \frac{a+\sqrt{ab}+b}{3}\,\sqrt{ab}$$

By finding the partial derivatives of $f(a,b)$ and with simple manipulation gives

$$a\frac{\partial f}{\partial a} - b\frac{\partial f}{\partial b} = (a-b)\left[-\frac{4ab}{(a+b)^3}(a^2+b^2) + \frac{4ab}{a+b} - \frac{\sqrt{ab}}{3}\right]$$

From Lemma 3,

$$(lna - lnb)\left(a\frac{\partial f}{\partial a} - b\frac{\partial f}{\partial b}\right) = -(lna - lnb)(a-b)\left[\frac{2H}{A}(C-A) + \frac{G}{3}\right] \leq 0$$

Holds for $a \geq b$.

This completes the proof of theorem 2.

Theorem 3. For $a, b > 0$, then the ratio of difference of means M_{CAGH} is schur geometrically concave.

Proof. Recall equation (12)

$$M_{CAGH}(\mathrm{a,b}) = \frac{C(a,b)-A(a,b)}{G(a,b)-H(a,b)}.$$

From Lemma 2, if $CH - AG < 0$ then M_{AH_eGH} is concave.
Consider,

$$f(a,b) = CH - AG = \frac{a^2+b^2}{a+b}\,\frac{2ab}{a+b} - \frac{a+b}{2}\sqrt{ab}$$

By finding the partial derivatives of $f(a,b)$ and with simple manipulation gives

$$a\frac{\partial f}{\partial a} - b\frac{\partial f}{\partial b} = (a-b)\left[-\frac{4ab}{(a+b)^3}(a^2+b^2) + \frac{4ab}{a+b} - \frac{\sqrt{ab}}{2}\right]$$

From Lemma 3,

$$(lna - lnb)\left(a\frac{\partial f}{\partial a} - b\frac{\partial f}{\partial b}\right) = -(lna - lnb)(a-b)\left[\frac{2H}{A}(C-A) + \frac{G}{2}\right] \le 0$$

Holds for $a \ge b$.

This completes the proof of theorem 3.

3. APPLICATIONS TO MEAN INEQUALITIES

This section concern with some inters related mean inequalities of Ratio of difference of means.

Theorem 5.

Let $0 \le a \le b$, if $\frac{1}{2} \le t \le 1$ or $0 \le t \le \frac{1}{2}$. Then

$$M_{AH_eGH}\left(\sqrt{ab},\sqrt{ab}\right) \ge M_{AH_eGH}(a^t b^{1-t}, b^t a^{1-t}) \ge M_{AH_eGH}(a,b) \tag{18}$$

Proof. From Lemma 4,

$$\left(\ln\sqrt{ab}\,,\ln\sqrt{ab,}\right) \prec (\ln(a^t b^{1-t})\,,\ln(b^t a^{1-t})) \prec (\ln a\,,\ln b)$$

And by Theorem 1 the ratio of difference of means

$$M_{AH_eGH}(\mathrm{a,b}) = \frac{A(a,b)-H_e(a,b)}{G(a,b)-H(a,b)} = \frac{a+b}{2}\,\frac{2ab}{a+b} - \frac{a+\sqrt{ab}+b}{3}\sqrt{ab}$$

$$= \frac{2ab-(a+b)\sqrt{ab}}{3}$$

Is Schur geometrically concave in R_+^2 so we have

$$M_{AH_eGH}\left(\sqrt{ab},\sqrt{ab}\right) \ge M_{AH_eGH}(a^t b^{1-t}, b^t a^{1-t}) \ge M_{AH_eGH}(a,b).$$

This completes the proof of theorem 5.

Theorem 6.

Let $0 \leq a \leq b$, if $\frac{1}{2} \leq t \leq 1$ or $0 \leq t \leq \frac{1}{2}$. Then

$$M_{CH-GH_e}\left(\sqrt{ab},\sqrt{ab}\right) \geq M_{CH-GH_e}(a^t b^{1-t}, b^t a^{1-t}) \geq M_{CH-GH_e}(a,b) \tag{19}$$

Proof. From Lemma 4,

$$\left(\ln\sqrt{ab}\,,\ln\sqrt{ab,}\right) \prec (\ln(a^t b^{1-t})\,,\ln(b^t a^{1-t})) \prec (\ln a\,,\ln b)$$

And by Theorem 1 the ratio of difference of means

$$M_{CH-GH_e}(\mathrm{a,b}) = \frac{C(a,b)-H_e(a,b)}{G(a,b)-H(a,b)} = \frac{a^2+b^2}{a+b}\,\frac{2ab}{a+b} - \frac{a+\sqrt{ab}+b}{3}\,\sqrt{ab}$$

Is Schur geometrically concave inR_+^2 so we have

$$M_{CH-GH_e}\left(\sqrt{ab},\sqrt{ab}\right) \geq M_{CH-GH_e}(a^t b^{1-t}, b^t a^{1-t}) \geq M_{CH-GH_e}(a,b)$$

This completes the proof of theorem 6.

Theorem 7.

Let $0 \leq a \leq b$, if $\frac{1}{2} \leq t \leq 1$ or $0 \leq t \leq \frac{1}{2}$. Then

$$M_{CAGH}\left(\sqrt{ab},\sqrt{ab}\right) \geq M_{CAGH}(a^t b^{1-t}, b^t a^{1-t}) \geq M_{CAGH}(a,b) \tag{20}$$

Proof. From Lemma 4,

$$\left(\ln\sqrt{ab}\,,\ln\sqrt{ab,}\right) \prec (\ln(a^t b^{1-t})\,,\ln(b^t a^{1-t})) \prec (\ln a\,,\ln b)$$

And by Theorem 1 the ratio of difference of means

$$M_{CAGH}(\mathrm{a,b}) = \frac{C(a,b)-A(a,b)}{G(a,b)-H(a,b)} = \frac{a^2+b^2}{a+b}\,\frac{2ab}{a+b} - \frac{a+b}{2}\,\sqrt{ab}$$

Is Schur geometrically concave in R_+^2 so we have

$$M_{CAGH}\left(\sqrt{ab},\sqrt{ab}\right) \geq M_{CAGH}(a^t b^{1-t}, b^t a^{1-t}) \geq M_{CAGH}(a,b)$$

This completes the proof of theorem 7.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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