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Oscillation Criteria for the Solutions of Higher Order Functional Difference Equations of Neutral type

Radhanath Rath*[1] and Ajit kumar Bhuyan[2] and B.L.S. Barik[3]

[1] *P.G Department of Mathematics, Khallikote Autonomous College, Berhampur, Odisha, India, 760001.*

[2] *Department of Mathematics., G.E.C, Bhubaneswar,Odisha, India.*

[3] *Department of Mathematics, K.I.S.T, Bhubaneswar,Odisha, India.*

Research Article

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Abstract

In this paper, necessary and sufficient conditions are obtained so that the neutral functional difference equation

$$\Delta^m\big(y_n - y_{\tau(n)}\big) + q_n G(y_{\sigma(n)}) = f_n, \quad n \geq n_0,$$

admits a positive bounded solution, where $m \geq 1$ is an odd integer, Δ is the forward difference operator given by $\Delta y_n = y_{n+1} - y_n$; $\{f_n\}$, $\{q_n\}$, are sequences of real numbers with $q_n \geq 0$, $G \in C(\mathbb{R}, \mathbb{R})$. The results of this paper improve and extend some recent work [6, 15].

Keywords: Asymptotic behavior, difference equation, oscillatory solution, non oscillatory solution.

2000 Mathematics Subject Classification: 39A10, 39A12.

1 Introduction

In this paper, necessary and sufficient conditions are obtained so that the neutral functional difference equation

$$\Delta^m\big(y_n - y_{\tau(n)}\big) + q_n G(y_{\sigma(n)}) = f_n, \quad n \geq n_0, \tag{1.1}$$

admits a positive bounded solution, where $m \geq 1$ is an odd integer, Δ is the forward difference operator given by $\Delta y_n = y_{n+1} - y_n$; $\{f_n\}$, $\{q_n\}$, are sequences of real numbers with $q_n \geq 0$. It is supposed that $G \in C(\mathbb{R}, \mathbb{R})$, is non-decreasing and $xG(x) > 0$. Moreover, it will be assumed that $\tau(n)$ and $\sigma(n)$ are increasing sequences of integers, such that they are less than n and approach $+\infty$ as $n \to \infty$.

All over the world, during the last decade or two a lot of research activity is undertaken on the study of the oscillation of neutral delay difference equations(NDDEs in short). For recent results and references see the monographs [1, 4, 5], the papers [2, 3], [6], [9]–[16] and the references cited therein. In these papers the authors have studied the oscillation and non-oscillation of solutions of the NDDE

$$\Delta(y_n - p_n y_{n-k}) + q_n G(y_{n-r}) = f_n \tag{1.2}$$

Corresponding author: E-mail: radhanathmath@yahoo.co.in

under the condition

$$\sum_{n=n_0}^{\infty} n^{m-1} q_n = \infty, \tag{1.3}$$

or the condition

$$\sum_{n=n_0}^{\infty} n^{m-2} q_n = \infty. \tag{1.4}$$

However, in this work we prove that the necessary and sufficient conditions for the oscillation of all bounded solutions of

$$\Delta^m\big(y_n - y_{\tau(n)}\big) + q_n G(y_{\sigma(n)}) = 0, \quad n \geq n_0, \tag{1.5}$$

is

$$\sum_{n=n_0}^{\infty} n^{m} q_n = \infty, \tag{1.6}$$

which is weaker than (1.3) and (1.4). Thus our results improve the following theorems, which are particular cases of some of the results in [6] and [14].

Theorem 1.1. *[6, Corollary 4.6] Let* (1.3) *holds. Further, assume that the following conditions hold.*

(H1) $\sigma(n)/n \geq \mu > 0$ *for all* $n \geq n_0$*;*

(H2) $\liminf_{|u|\to\infty} \frac{G(u)}{u} \geq \delta > 0$.

Then every bounded solution of (1.5) *oscillates.*

Theorem 1.2. *[6, Theorem 4.9] Let* $\tau(n) = n - k$ *for some* k*. Assume that (H1), (H2) hold. Further assume the following condition.*

(H3) Suppose that for every subsequence $\{q_{n_j}\}$ *of* $\{q_n\}$*, we have*

$$\sum_{j=0}^{\infty} (n_j)^{m-1} q_{n_j} = \infty,$$

or equivalently $\liminf_{n\to\infty} n^{m-1} q_n > 0$.

Then every bounded solution of (1.5) *oscillates or tends to zero as* $n \to \infty$.

Note that (H3) implies (1.3).

Theorem 1.3. *[14] Suppose that* m *is odd and the following condition holds.*

$$\sum_{n=n_0}^{\infty} q_n = \infty.$$

Then every bounded solution of (1.5) *oscillates.*

Let n_0 be a fixed nonnegative integer. Let $\rho = \min\{\tau(n_0), \sigma(n_0)\}$. By a solution of (1.1) we mean a real sequence $\{y_n\}$ which is defined for all integers $n \geq \rho$ and satisfies (1.1) for $n \geq n_0$. Clearly if the initial condition

$$y_n = a_n \quad \text{for} \quad \rho \leq n \leq n_0 + m - 1, \tag{1.7}$$

is given then the equation (1.1) has a unique solution satisfying the given initial condition (1.7). A solution $\{y_n\}$ of (1.1) is said to be oscillatory if for every positive integer $n_0 > 0$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

2 SOME LEMMAS

For our main results we need the following definitions and lemmas.

Definition 2.1. *For any positive integer $n \geq n_0$, define*

$$\tau_{-1}(n) = \{m : m \text{ is an integer } \geq n \text{ and } \tau(m) = n\}.$$

Remark 2.2. The function τ_{-1} defined above is the inverse function of $\tau(n)$. Since $\tau(n)$ is increasing, it is one-one. If n is a positive integer greater than or equal to n_0 then $\tau_{-1}(\tau(n)) = n$.

Definition 2.3. Define

$$\tau^0_{-1}(n) = n, \tau^1_{-1}(n) = \tau_{-1}(n), \tau^2_{-1}(n) = \tau_{-1}(\tau_{-1}(n)).$$

For any positive integer $i > 2$ we define

$$\tau^i_{-1}(n) = \tau_{-1}(\tau^{i-1}_{-1}(n)).$$

Definition 2.4. Define the factorial function(See[7, page-20]) by

$$n^{(k)} := n\,(n-1)\ldots(n-k+1)\,,$$

where $k \leq n$ and $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Note that $n^{(k)} = 0$, if $k > n$.

Lemma 2.5. *[6, Lemma 2.4] Let $p \in \mathbb{N}$ and $x\,(n)$ be a non oscillatory sequence which is positive for large n. If there exists an integer $p_0 \in \{0, 1, \ldots, p-1\}$ such that $\Delta^{p_0} w\,(\infty)$ exits(finite) and $\Delta^i w\,(\infty) = 0$ for all $i \in \{p_0+1, \ldots, p-1\}$. Then*

$$\Delta^p w\,(n) = -x\,(n) \tag{2.1}$$

implies

$$\Delta^{p_0} w\,(n) = \Delta^{p_0} w\,(\infty) + \frac{(-1)^{p-p_0-1}}{(p-p_0-1)!} \sum_{i=n}^{\infty} (i+p-p_0-1-n)^{(p-p_0-1)}\, x\,(i) \tag{2.2}$$

for all sufficiently large n.

Lemma 2.6. *[1] Let z_n be a real valued function defined for $n \in N(n_0) = \{n_0, n_0+1, ...\}, n_0 \geq 0$ and $z_n > 0$ with $\Delta^m z_n$ of constant sign on $N(n_0)$ and not identically zero.Then there exists an integer p_0, $0 \leq p_0 \leq m-1$, with $m + p_0$ odd for $\Delta^m z_n \leq 0$ and $(m + p_0)$ even for $\Delta^m z_n \geq 0$, such that*

$$\Delta^i z_n > 0 \quad \textit{for} \quad n \geq n_0, \quad 0 \leq i \leq p_0,$$

and

$$(-1)^{p_0+i} \Delta^i z_n > 0, \quad \textit{for} \quad n \geq n_0, \quad p_0 + 1 \leq i \leq m - 1.$$

Lemma 2.7. *[8] If $\sum u_n$ and $\sum v_n$ are two positive term series such that $\lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = l$, where l is a non-zero finite number, then the two series converge or diverge together. If $l = 0$ then convergence of $\sum v_n$ implies the convergence of $\sum u_n$. If $l = \infty$ then divergence of $\sum v_n$ implies the divergence of $\sum u_n$.*

Lemma 2.8 (Schauder Fixed Point Theorem [5])**.** *Let S be a closed, convex and nonempty subset of a Banach space X. Let $B : S \to S$ be a continuous mapping such that $B(S)$ be a relatively compact subset of X. Then B has at least one fixed point in S. This means there is an $x \in S$ such that $Bx = x$.*

Further, the following Lemma, that can be easily proved, generalizes [9, Lemma 2.1].

Lemma 2.9. *Let $\{f_n\}$ and $\{g_n\}$ be sequences of real numbers for $n \geq 0$ such that*

$$f_n = g_n - pg_{\tau(n)}, \quad n \geq n_0$$

where $p \in \mathbb{R}, p \neq 1$ and $\tau(n) \leq n, \forall n$, with $\lim_{n\to\infty} \tau(n) = \infty$. Suppose that $\lim_{n\to\infty} f_n = \lambda \in \mathbb{R}$ exists. Then the following statements hold.

(i) If $\liminf_{n\to\infty} g_n = a \in \mathbb{R}$ then $\lambda = (1-p)a$.

(ii) If $\limsup_{n\to\infty} g_n = b \in \mathbb{R}$, then $\lambda = (1-p)b$.

Remark 2.10. In the above lemma, if $p = 1$ then (see [5, Corollary 1.5.1, page 19]) $\lambda = 0$.

3 MAIN RESULTS

We need the following two assumptions for our results in this section.

$$|\sum_{i=1}^{\infty} \sum_{j=\tau_{-1}^{i}(n_0)}^{\infty} j^{m-1} f_j| < \infty. \tag{3.1}$$

$$\sum_{i=1}^{\infty} \sum_{j=\tau_{-1}^{i}(n_0)}^{\infty} j^{m-1} q_j < \infty. \tag{3.2}$$

Remark 3.1. We may recall the well known factorial function $n^{(r)} = (n-1)(n-2)...(n-r+1)$, if $r \leq n$, otherwise it is zero. Since $(n-r+1)^r < n^{(r)} < n^r$, then from Lemma 2.7, it follows that (3.2) implies and implied by the condition

$$|\sum_{i=1}^{\infty} \sum_{j=\tau_{-1}^{i}(n_0)}^{\infty} (j - \tau_{-1}^{i}(n_0) + m - 1)^{(m-1)} q_j| < \infty \tag{3.3}$$

and (3.1) implies and implied by

$$|\sum_{i=1}^{\infty} \sum_{j=\tau_{-1}^{i}(n_0)}^{\infty} (j - \tau_{-1}^{i}(n_0) + m - 1)^{(m-1)} f_j| < \infty. \tag{3.4}$$

Our first result reads as follows.

Theorem 3.2. *Suppose that for each positive integer $n \geq n_0$, $f_n \leq 0$, and that* (3.1) *holds. Then* (1.1) *admits a positive bounded solution if and only if* (3.2) *holds.*

Proof. Assume that (3.2) holds. We show that (1.1) admits a positive bounded solution. Using the continuity of G, we set

$$\mu = \max\{|G(x)| : 2 \leq x \leq 6\}. \tag{3.5}$$

Then by (3.2) and (3.1) and Remark 3.1, we find a positive integer $N_1 \geq n_0$ such that $n \geq N_1$ implies

$$|\frac{\mu}{(m-1)!} \sum_{i=1}^{\infty} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} (j - \tau_{-1}^{i}(n) + m - 1)^{(m-1)} q_j| < 1 \tag{3.6}$$

and

$$|\frac{\mu}{(m-1)!}\sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty}(j-\tau_{-1}^{i}(n)+m-1)^{(m-1)}f_j|<1. \tag{3.7}$$

Choose $N_2 \geq N_1$ such that $k \geq N_1$, where $k = \min\{\tau(N_2), \sigma(N_2)\}$. Let $X = l_\infty^{N_1}$, the Banach space of bounded real sequences $x = \{x_n\}$, with the supremum norm

$$\|x\| = \sup\{|x_n| : n \geq N_1\}.$$

In this space, we define the closed and convex set

$$S = \{y \in X : 2 \leq y_n \leq 6, n \geq N_1\}. \tag{3.8}$$

Now we define the operator B, from S to X, such that fixed points of B are solutions of (1.1). For $y \in S$, define

$$(By)_n = \begin{cases} (By)_{N_2}, & N_1 \leq n \leq N_2, \\ \frac{(-1)^m}{(m-1)!}\sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty}(j-\tau_{-1}^{i}(n)+m-1)^{(m-1)}q_jG(y_{\sigma(j)}) & \\ +\frac{(-1)^{m-1}}{(m-1)!}\sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty}(j-\tau_{-1}^{i}(n)+m-1)^{(m-1)}f_j+4, & n \geq N_2\,. \end{cases}$$

For $y = \{y_n\} \in S$, we have $(By)_n \leq 6$ and $(By)_n \geq 2$. Hence, $By \in S$. Then using (3.3) and (3.4) and proceeding as in the proof of [6, Theorem 5.4] we prove BS is relatively compact. By Lemma 2.8, there is a fixed point y^0 in S such that $By_n^0 = y_n^0$, for $n \geq N_2$, writing y_n for y_n^0 we obtain

$$\begin{aligned} y_n =& \frac{(-1)^m}{(m-1)!}\sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty}(j-\tau_{-1}^{i}(n)+m-1)^{(m-1)}q_jG(y_{\sigma(j)}) \\ &+\frac{(-1)^{m-1}}{(m-1)!}\sum_{i=1}^{\infty}\sum_{j=\tau_{-1}^{i}(n)}^{\infty}(j-\tau_{-1}^{i}(n)+m-1)^{(m-1)}f_j+4. \end{aligned}$$

For $n \geq N_2$, it follows that

$$\begin{aligned} y_n - y_{\tau(n)} =& \frac{(-1)^{m-1}}{(m-1)!}\sum_{j=n}^{\infty}(j-n+m-1)^{(m-1)}q_jG(y_{\sigma(j)}) \\ &+\frac{(-1)^m}{(m-1)!}\sum_{j=n}^{\infty}(j-n+m-1)^{(m-1)}f_j \end{aligned}$$

Applying Δ to both sides of the above equation for m times, we arrive at (1.1). This solution is bounded below by 2 which is a positive constant.

Conversely assume that (1.1) admits a positive bounded solution $\{y_n\}$. Then we find a positive integer n_0 such that $n \geq n_0$ implies

$$y_n > 0,\ y_{\tau(n)} > 0,\ y_{\sigma(n)} > 0.$$

Note that (3.4) follows from (3.1) by Remark 3.1. If we set

$$F_n = \frac{(-1)^m}{(m-1)!}\sum_{j=n}^{\infty}(j-n+m-1)^{(m-1)}f_j$$

then $\Delta^m F_n = f_n$. Since m is odd and $f_n \leq 0$,

$$F_n \geq 0 \quad \text{and} \quad \lim_{n\to\infty} F_n = 0, \tag{3.9}$$

due to (3.4). Setting

$$z_n = y_n - y_{\tau(n)} \quad \text{and} \quad w_n = z_n - F_n \tag{3.10}$$

for $n \geq n_0$, we obtain

$$\Delta^m w_n = -q_n G(y_{\sigma(n)}) \leq 0. \tag{3.11}$$

Since w_n is bounded then $\lim_{n\to\infty} w_n = l$ exists. From (3.9) and (3.10), we obtain $\lim_{n\to\infty} z_n = l$. From Lemma 2.9 and the following remark, it follows that $l = 0$. Here, in this case $p_0 = 0$ by Lemma 2.6. Then $(-1)^i \Delta^i w_n > 0$ for $1 \leq i \leq m-1$. Hence $w_n > 0$ for $n \geq n_1 \geq n_0$ as it is decreasing. From (3.10) we obtain $y_n > y_{\tau(n)}$ for $n \geq n_1$ because $F_n > 0$. This implies $\liminf_{n\to\infty} y_n > 0$. Thus there exists $\gamma > 0$ such that $y_n > \gamma$ for $n \geq n_2 \geq n_1$. Applying Lemma 2.5 to (3.11) (Here $p_0 = 0$) we obtain for $n \geq n_3 \geq n_2$,

$$w_n = \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i G(y_{\sigma(i)}).$$

Then it follows that

$$y_{\tau(n)} < y_n - \frac{G(\gamma)}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i + \frac{1}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} f_i. \tag{3.12}$$

Replacing n by $\tau_{-1}(n)$ in (3.12), we get

$$\begin{aligned} y_n <& y_{\tau_{-1}(n)} - \frac{G(\gamma)}{(m-1)!} \sum_{i=\tau_{-1}(n)}^{\infty} (i - \tau_{-1}(n) + m - 1)^{(m-1)} q_i \\ &+ \frac{1}{(m-1)!} \sum_{i=\tau_{-1}(n)}^{\infty} (i - \tau_{-1}(n) + m - 1)^{(m-1)} f_i. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13) it follows that,

$$\begin{aligned} y_{\tau(n)} <& y_{\tau_{-1}(n)} - \frac{G(\gamma)}{(m-1)!} \sum_{i=0}^{1} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} (j - \tau_{-1}^{i}(n) + m - 1)^{(m-1)} q_j \\ &+ \frac{1}{(m-1)!} \sum_{i=0}^{1} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} (j - \tau_{-1}^{i}(n) + m - 1)^{(m-1)} f_j. \end{aligned} \tag{3.14}$$

Hence repeating the above process k times, we obtain

$$\begin{aligned} y_{\tau(n)} <& y_{\tau_{-1}^{k}(n)} - \frac{G(\gamma)}{(m-1)!} \sum_{i=0}^{k} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} (j - \tau_{-1}^{i}(n) + m - 1)^{(m-1)} q_j \\ &+ \frac{1}{(m-1)!} \sum_{i=0}^{k} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} (j - \tau_{-1}^{i}(n) + m - 1)^{(m-1)} f_j. \end{aligned} \tag{3.15}$$

Hence

$$\begin{aligned} &\frac{G(\gamma)}{(m-1)!} \sum_{i=0}^{k} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} (j - \tau_{-1}^{i}(n) + m - 1)^{(m-1)} q_j \\ &< y_{\tau_{-1}^{k}(n)} - y_{\tau(n)} + \frac{1}{(m-1)!} \sum_{i=0}^{k} \sum_{j=\tau_{-1}^{i}(n)}^{\infty} (j - \tau_{-1}^{i}(n) + m - 1)^{(m-1)} f_j. \end{aligned} \tag{3.16}$$

Taking limit $k \to \infty$, using (3.4) and that y_n is bounded, we obtain (3.3), from which (3.2) follows by Remark 3.1. Thus, the proof of the theorem is complete.

□

The following corollary could be proved, proceeding as in the proof of the above theorem.

Corollary 3.3. *Suppose that $f_n \geq 0$, for each positive integer $n \geq n_0$ and* (3.1) *holds. Then* (1.1) *admits a negative bounded solution if and only if* (3.2) *holds.*

Next, our objective is to prove a theorem which shows that (3.2) is equivalent to the condition (1.6)

Theorem 3.4. *Consider the delay difference equation*

$$\Delta^{m+1} x_n + q_n x_{\sigma(n)} = 0, \quad n > 0. \tag{3.17}$$

Then the following conditions are equivalent.

(a) Every bounded solution of (3.17) *oscillates.*

(b) The condition (1.6) *holds.*

(c) The condition

$$\sum_{i=0}^{\infty} \sum_{j=n_0+ik}^{\infty} j^{m-1} q_j = \infty, \tag{3.18}$$

holds for any fixed positive integer k and $n_0 > 0$.

Proof. We show that $(a) \Leftrightarrow (c)$ and $(a) \Leftrightarrow (b)$. Hence $(b) \Leftrightarrow (c)$. First let us prove $(a) \Leftrightarrow (c)$. Suppose that (a) holds. For the sake of contradiction, assume that (c) does not hold. Then

$$\sum_{i=0}^{\infty} \sum_{j=n_0+ik}^{\infty} j^{m-1} q_j < \infty.$$

Hence we can find an integer $n_1 > 0$, large enough such that

$$\frac{k}{(m-1)!} \sum_{i=n_1}^{\infty} \sum_{j=n_0+ik}^{\infty} j^{m-1} q_j < 1/3. \tag{3.19}$$

Let $n_2 = n_0 + n_1 k$. Then from (3.19), we obtain

$$\frac{k}{(m-1)!} \sum_{i=0}^{\infty} \sum_{j=n+ik}^{\infty} j^{m-1} q_j < 1/3 \quad \text{for} \quad n \geq n_2.$$

Using Remark 3.1, we obtain

$$\frac{k}{(m-1)!} \sum_{i=0}^{\infty} \sum_{j=n+ik}^{\infty} (j-n-ik+m-1)^{(m-1)} q_j < 1/3 \tag{3.20}$$

for $n \geq n_2$. Choose $N_0 \geq n_2$ and $N_1 > N_0$ such that $\sigma(N_1) \geq N_0$. Let $X = l_\infty^{N_0}$ be the Banach space of bounded real sequences $x = \{x_n\}, n \geq N_0$ with supremum norm $||x|| = \sup\{|x_n| : n \geq N_0\}$. Define S to be a closed subset of X such that $S = \{y \in X : 1 \leq y_n \leq 3/2, n \geq N_0\}$. Then S is a metric space , where the metric is induced by the norm on X. For $x \in S$, define

$$Ax_n = \begin{cases} 1, & N_0 \leq n \leq N_1, \\ 1 + \frac{1}{(m-1)!} \sum_{i=N_1}^{n-1} \sum_{j=i}^{\infty} (j-i+m-1)^{(m-1)} q_j x_{\sigma(j)}, & n \geq N_1. \end{cases}$$

Then for $n \geq N_1$,

$$
\begin{aligned}
1 \leq Ax_n &< 1 + \frac{1}{(m-1)!} \sum_{i=N_1}^{\infty} \sum_{j=i}^{\infty} (j-i+m-1)^{(m-1)} q_j x_{\sigma(j)} \\
&\leq 1 + \frac{1}{(m-1)!} \sum_{p=0}^{\infty} \sum_{i=N_1+pk}^{N_1+pk+k-1} \sum_{j=i}^{\infty} (j-i+m-1)^{(m-1)} q_j x_{\sigma(j)} \\
&\leq 1 + \frac{k}{(m-1)!} \sum_{p=0}^{\infty} \sum_{j=N_1+pk}^{\infty} (j-N_1-pk+m-1)^{(m-1)} q_j x_{\sigma(j)} \\
&\leq 1 + \frac{3k}{2(m-1)!} \sum_{p=0}^{\infty} \sum_{j=N_1+pk}^{\infty} (j-N_1-pk+m-1)^{(m-1)} q_j \\
&\leq 1 + 1/2 \leq 3/2.
\end{aligned}
$$

Hence $AS \subset S$. Further, it may be shown that, for $x, y \in S, ||Ax - Ay|| \leq \frac{1}{3}||x - y||$. Hence A is a contraction. Consequently A has a unique fixed point x in S. It is a positive bounded solution of (3.17) for $n \geq N_2$, a contradiction. Hence $(a) \Rightarrow (c)$ holds.

Next, suppose that (c) holds. Let $x = \{x_n\}$ be a bounded non-oscillatory solution of (3.17). We may take, without any loss of generality $x_n > 0$, $x_{\sigma(n)} > 0$ for $n \geq n_0 > 0$. Then

$$\Delta^{m+1} x_n = -q_n x_{\sigma(n)} \leq 0 \tag{3.21}$$

for $n \geq n_1 \geq n_0$. Hence $x_n, \Delta x_n, ..., \Delta^m x_n$ are monotonic and are of constant sign for $n \geq n_2 \geq n_1$. Since x_n is bounded and m is odd, $\Delta x_n > 0$, by Lemma 2.6. This implies $\{x_n\}$ is non- decreasing. Since $x_n > 0$, we find $\alpha > 0$ such that $x_n > \alpha > 0$ for $n \geq n_3 \geq n_2$. Applying Lemma 2.5 to (3.21), (Here $p_0 = 1$) we obtain

$$
\begin{aligned}
\Delta x_n &= \lim_{n\to\infty} \Delta x_n + \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i x_{\sigma(i)} \\
&\geq \frac{(-1)^{m-1}}{(m-1)!} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i x_{\sigma(i)}
\end{aligned} \tag{3.22}
$$

Hence taking summation on both sides of the above equation we obtain

$$\sum_{p=0}^{j} \sum_{n=n_3+pk}^{n_3+pk+k-1} \Delta x_n \geq \frac{1}{(m-1)!} \sum_{p=0}^{j} \sum_{n=n_3+pk}^{n_3+pk+k-1} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i x_{\sigma(i)}. \tag{3.23}$$

This implies

$$
\begin{aligned}
x_{n_3+(j+1)k} - x_{n_3} &\geq \frac{1}{(m-1)!} \sum_{p=0}^{j} \sum_{n=n_3+pk}^{n_3+pk+k-1} \sum_{i=n}^{\infty} (i-n+m-1)^{(m-1)} q_i x_{\sigma(i)} \\
&\geq \frac{\alpha k}{(m-1)!} \sum_{p=0}^{j} \sum_{i=n_3+pk+k-1}^{\infty} (i-n_3-pk-k+m)^{(m-1)} q_i.
\end{aligned}
$$

Taking the limit $j \to \infty$ and using the fact that $\{x_n\}$ is bounded, we obtain

$$\sum_{p=0}^{\infty} \sum_{i=n_3+pk+k-1}^{\infty} (i-n_3-pk-k+m)^{(m-1)} q_i < \infty,$$

a contradiction. Hence $(c) \Rightarrow (a)$ is proved.

Next, to show $(a) \Rightarrow (b)$. Suppose that (a) holds. For the sake of contradiction, assume (b) does not hold. That is $\sum_{i=n_0}^{\infty}(i-n_0+m)^{(m)}q_i < \infty$. Hence for any $n \geq n_0$, we have $\frac{1}{m!}\sum_{i=n}^{\infty}(i-n+m)^{(m)}q_i < \infty$. Then proceeding as in the proof of the case $(a) \Rightarrow (c)$, we find N_0 such that $n \geq N_0$ implies $\frac{1}{m!}\sum_{i=n}^{\infty}(i-n+m)^{(m)}q_i < 1/4$. Let $N_1 > N_0$ such that $\sigma(N_1) \geq N_0$. Set $S = \{x_n \in X : 3/4 \leq x_n \leq 1, n \geq N_0\}$ and for $x \in S$, define

$$Ax_n = \begin{cases} Ax_{N_1}, & N_0 \leq n \leq N_1, \\ 1 - \frac{1}{m!}\sum_{i=n}^{\infty}(i-n+m)^{(m)}q_i x_{\sigma(i)}, & n \geq N_1. \end{cases}$$

Clearly, $3/4 \leq A(x_n) \leq 1$. Hence $A(S) \subset S$ and A is a contraction. Hence A has a unique fixed point in S which is a positive bounded solution of (3.17), a contradiction. Hence (a) implies (b).

Next, suppose (b) holds. Let $\{x_n\}$ be a bounded non-oscillatory solution of (3.17) for $n \geq n_0 > 0$. Proceeding as in the proof of the case $(c) \Rightarrow (a)$, we obtain $\lim_{n\to\infty} x_n = l > 0$ exists. By Lemma 2.6 we have $p_0 = 1$. Then due to the boundedness of x_n it follows that $\lim_{n\to\infty}\Delta x_n = 0$. From (3.17), using Lemma 2.5, for $p = m+1$ and $p_0 = 0$, we get

$$x_n = l + \frac{(-1)^m}{m!}\sum_{i=n}^{\infty}(i-n+m)^{(m)}q_i x_{\sigma(i)}.$$

This implies

$$\left|\frac{(-1)^m}{m!}\sum_{i=n}^{\infty}(i-n+m)^{(m)}q_i x_{\sigma(i)}\right| < \infty.$$

On the other hand,

$$\frac{1}{m!}\sum_{i=n}^{\infty}(i-n+m)^{(m)}q_i x_{\sigma(i)} > \frac{l}{2(m!)}\sum_{i=n}^{\infty}(i-n+m)^{(m)}q_i = \infty.$$

a contradiction. Hence $(b) \Rightarrow (a)$. Thus the theorem is completely proved. □

Remark 3.5. If $\tau(n) = n-k$ then $\tau_{-1}(n) = n+k$ for some k, and $\tau_{-1}^{i}(n) = n+ik$. Using this in (3.2), and further using Theorem 3.4, one may conclude that (3.2) $\Leftrightarrow$ (1.6).

From Remark 3.5 and Theorem 3.2 the following result follows.

Theorem 3.6. *Every bounded solution of* (1.5) *with* $\tau(n) = n-k$ *for some* k*, oscillates if and only if* (1.6) *holds.*

Remark 3.7. The above theorem improves theorems 1.1, 1.2 and 1.3.

At the end we would like to present an example which illustrates our results and establishes the importance of this work.

Example
Consider the neutral difference equation

$$\Delta^3(y_n - y_{n-1}) + q_n G(y_{n-2}) = 0, \tag{3.24}$$

where $G(u) = u^{\frac{1}{5}}$. Consider

$$y_n = \begin{cases} \frac{1}{n^5}; & n \text{ is odd} \\ -\frac{1}{n^5}; & n \text{ is even.} \end{cases}$$

Then

$$\Delta^3(y_n - y_{n-1}) = y_{n+3} - 4y_{n+2} + 6y_{n+1} - 4y_n + y_{n-1}.$$

If n is odd then

$$\begin{aligned}\Delta^3(y_n - y_{n-1}) &= -\frac{1}{(n+3)^5} - \frac{4}{(n+2)^5} - \frac{6}{(n+1)^5} - \frac{4}{n^5} - \frac{1}{(n-1)^5} \\ &= -\frac{16n^{20} + ...}{n^{25} + ...} \\ &= -A.\end{aligned}$$

$G(y_{n-2}) = \frac{1}{n-2}$. On the other hand if n is even then

$$\Delta^3(y_n - y_{n-1}) = A \quad \text{and} \quad G(y_{n-2}) = -\frac{1}{n-2}.$$

In either case, we take

$$q_n = (n-2)A \approx \frac{1}{n^4} \quad \text{for large} \quad n.$$

Note that q_n satisfies the condition (1.6) of this paper. Clearly, $\{y_n\}$ is a bounded solution of the (3.24) which oscillates. However, it does not satisfy any one of the conditions (1.3), (1.4)or (H3). Hence the results of [6, 15] or any other paper in the reference cannot be applied to the neutral equation (3.24).

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Competing Interests

The authors declare that no competing interests exist.

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