British Journal of Mathematics & Computer Science
3(3): 352-389, 2013

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Lossless Transmission Lines Terminated by *L*-Load in Series Connected to Parallel Connected *GL*-Loads

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Research Article

Received: 08 March 2013
Accepted: 03 May 2013
Published: 15 May 2013

Abstract

The paper analyses transmission lines terminated by nonlinear loads situated as follows: *GC*-loads are connected parallel and *L*-load is connected in series to them (Fig. 1). First, we formulate boundary conditions for a lossless transmission line system on the basis of Kirchhoff's law. Then, we reduce the mixed problem for the system in question to initial value problem for a neutral system on the boundary. We introduce an operator in a suitable function space whose fixed point is a periodic solution of the neutral system. The obtained conditions are easily verifiable. We demonstrate the advantages of our method in Numerical example.

Keywords: *Lossless transmission line, Mixed problem for hyperbolic system, Neutral equation, Periodic solution, Fixed point theorem, Kirchhoff's law.*

1 Introduction

The transmission line theory is based on the Telegrapher equations, which, from a mathematical point of view, present a first order hyperbolic system of partial differential equations with unknown functions the voltage and the current. The subject of transmission lines has grown in importance because of the many applications [1-9].

In our previous papers we have considered lossless and lossy transmission lines terminated by various configurations of nonlinear loads – connected in series, parallel connected, and so on [10-16]. The main purpose of the present paper is to consider a lossless transmission line terminated by nonlinear *RGCL*-loads placed in the following way: *GC*-loads are parallel and connected in series to *L*-load (Fig. 1). Such ~~a~~ configuration arises when considering the nature of the self-excitation of a tube generator containing a self-bias element in a grid circuit [17]. Such ~~a~~ configuration is also a basic element of a diode detection circuit [17,18].

The first difficulty is to derive the boundary conditions as a consequence of Kirchhoff's law (Fig. 1) and to formulate the mixed problem for the hyperbolic system. The second one is to reduce the

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mixed problem for the hyperbolic system to an initial value problem for neutral equations on the boundary. We overcome the last difficulty using the basic result [19] and the recent result [20].

Finally, we have to introduce a suitable operator whose fixed point is a periodic solution to the problem stated. By means of the fixed point method [21], we obtain an existence-uniqueness of the periodic solution.

The paper consists of six subsections. In Subsection 2.1, on the basis of Kirchhoff's law, we derive the boundary conditions and then formulate the mixed problem for the hyperbolic system or the transmission line system. In Subsection 2.2, we reduce the mixed problem to an initial value problem on the boundary. In Subsection 2.3, we analyse the arising nonlinearities and make some estimates which we use further on. In Subsection 2.4, we introduce an operator presentation of the periodic problem. The key role is played by Lemma 4.3: the operator has a fixed point iff the neutral system has a periodic solution. Subsection 2.5 contains some technical preliminaries, namely Lipschitz estimates of the right hand sides of the equations. Subsection 2.6 contains the main result − existence-uniqueness theorem for a periodic solution of the neutral system. Finally, in Section 3, we demonstrate how to apply our results to specific problems verifying only a system of inequalities.

2 Main Results

2.1 Derivation of the Boundary Conditions and Formulation of the Mixed Problem

Let Λ be the length of the transmission line and $T = \Lambda/(1/\sqrt{LC}) = \Lambda\sqrt{LC}$, where L is per unit-length inductance and C – per unit-length capacitance.

In accordance to Kirchhoff's *V*-law (Fig. 1), we have to collect the currents of the elements G_p and C_p and after that to collect the voltage of G_pC_p with the voltage of L_p $(p = 0,1)$.

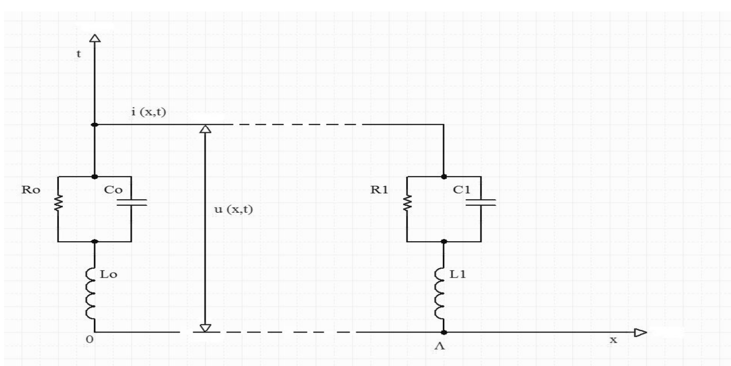

Fig. 1. Lossless transmission line terminated by nonlinear loads at both ends

We assume that G_p, C_p and L_p are nonlinear elements, that is, $G_p = G_p(u), L_p = L_p(i)$, $C_p = C_p(u)$ are nonlinear functions, and $\tilde{C}_p(u_{GpCp}) = u_{GpCp} C_p(u_{GpCp})$. So we have

$$i_{Gp} = G_p(u_{GpCp}),\ \ i_{Cp} = \frac{d\left(u_{GpCp} C_p(u_{GpCp})\right)}{dt} \equiv \frac{d\tilde{C}_p(u_{GpCp})}{dt},$$
$$i_{GpCp} = i_{G_p} + i_{C_p} = G_p(u_{G_pC_p}) + \frac{d\tilde{C}_p(u_{GpCp})}{dt} = G_p(u_{GpCp}) + \frac{d\left(u_{GpCp} C_p(u_{GpCp})\right)}{dt} =$$
$$= G_p(u_{GpCp}) + \left[u_{GpCp} \frac{dC_p(u_{GpCp})}{du_{GpCp}} + C_p(u_{GpCp})\right]\frac{du_{GpCp}}{dt} \equiv G_p(u_{GpCp}) + \frac{d\tilde{C}_p(u_{GpCp})}{du_{GpCp}} \frac{du_{GpCp}}{dt}.$$

For the left end we have $i_{G_0C_0}(t) = i(0,t),\ u_{G_0C_0}(t) = u(0,t)$ and therefore

$$\left[u_{G_0C_0} \frac{dC_0(u_{G_0C_0})}{du_{G_0C_0}} + C_0(u_{G_0C_0})\right]\frac{du_{G_0C_0}}{dt} = i(0,t) - G_0(u_{G_0C_0}).$$

Kirchhoff's voltage-law yields

$$u(0,t) = u_{L_0} + u_{G_0C_0}(t) - E_0(t) \tag{1.1}$$

and then, in view of

$$u_{L_0} = \frac{d\tilde{L}_0(i_{G_0C_0})}{dt} = \frac{d(L_0(i_{G_0C_0}).i_{G_0C_0})}{dt},$$

we obtain

$$\left[i(0,t)\frac{dL_0(i(0,t))}{di_{G_0C_0}} + L_0(i(0,t))\right]\frac{di(0,t)}{dt} = u(0,t) - u_{G_0C_0}(t) + E_0(t).$$

Finally, we have to solve the system

$$\left[u_{G_0C_0} \frac{dC_0(u_{G_0C_0})}{du_{G_0C_0}} + C_0(u_{G_0C_0})\right]\frac{du_{G_0C_0}}{dt} = i(0,t) - G_0(u_{G_0C_0}),$$
$$\left[i(0,t)\frac{dL_0(i(0,t))}{di_{G_0C_0}} + L_0(i(0,t))\right]\frac{di(0,t)}{dt} = u(0,t) - u_{G_0C_0}(t) + E_0(t).$$

For the right end we have $i_{G_1C_1}(t) = i(\Lambda,t),\ u_{G_1C_1}(t) = u(\Lambda,t)$ and therefore

$$\left[u_{G_1C_1}\frac{dC_1(u_{G_1C_1})}{du_{G_1C_1}}+C_1(u_{G_1C_1})\right]\frac{du_{G_1C_1}}{dt}=-i(\Lambda,t)+G_1(u_{G_1C_1})\ ,$$
$$\left[i(\Lambda,t)\frac{dL_1(i(\Lambda,t))}{di_{G_1C_1}}+L_1(i(\Lambda,t))\right]\frac{di(\Lambda,t)}{dt}=-u(\Lambda,t)+u_{G_1C_1}(t)-E_1(t).$$

Here $E_p(t),(p=0,1)$ are source functions connected in series to $RGCL$ -loads.

Now we are able to formulate the initial-boundary value problem (or mixed problem) for the hyperbolic transmission line equations: to find a solution $(u(x,t),\, i(x,t))$ to the first order partial differential system of hyperbolic type

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t}+L\frac{\partial i(x,t)}{\partial x}=0,\\&\frac{\partial i(x,t)}{\partial t}+C\frac{\partial u(x,t)}{\partial x}=0\end{aligned}\tag{1.2}$$

for $(x,t)\in\Pi=\{(x,t)\in R^2:0\le x\le\Lambda,\ t\ge 0\}$, satisfying the initial conditions

$$u(x,0)=u_0(x),\ i(x,0)=i_0(x)\ \text{for}\ x\in[0,\Lambda]\tag{1.3}$$

and the boundary conditions for $x=0$:

$$\begin{aligned}&\left[u_{G_0C_0}\frac{dC_0(u_{G_0C_0})}{du_{G_0C_0}}+C_0(u_{G_0C_0})\right]\frac{du_{G_0C_0}}{dt}=i(0,t)-G_0(u_{G_0C_0}),\\&\left[i(0,t)\frac{dL_0(i(0,t))}{di_{G_0C_0}}+L_0(i(0,t))\right]\frac{di(0,t)}{dt}=u(0,t)-u_{G_0C_0}(t)+E_0(t)\end{aligned}\tag{1.4}$$

and for $x=\Lambda$:

$$\begin{aligned}&\left[u_{G_1C_1}\frac{dC_1(u_{G_1C_1})}{du_{G_1C_1}}+C_1(u_{G_1C_1})\right]\frac{du_{G_1C_1}}{dt}=i(\Lambda,t)-G_1(u_{G_1C_1}),\\&\left[i(\Lambda,t)\frac{dL_1(i(\Lambda,t))}{di_{G_1C_1}}+L_1(i(\Lambda,t))\right]\frac{di(\Lambda,t)}{dt}=-u(\Lambda,t)+u_{G_1C_1}(t)-E_1(t).\end{aligned}\tag{1.5}$$

2.2 Reducing the Mixed Problem to an Initial Value Problem on the Boundary

We proceed from the lossless transmission line, that is, from the system:

$$\frac{\partial u(x,t)}{\partial x}+L\frac{\partial i(x,t)}{\partial t}=0,$$
$$\frac{\partial i(x,t)}{\partial x}+C\frac{\partial u(x,t)}{dt}=0. \qquad (2.1)$$

Rewrite system (2.1) in the form

$$\frac{\partial u(x,t)}{\partial t}+\frac{1}{C}\frac{\partial i(x,t)}{\partial x}=0,$$
$$\sqrt{\frac{L}{C}}\frac{\partial i(x,t)}{\partial t}+\sqrt{\frac{1}{LC}}\frac{\partial u(x,t)}{\partial x}=0. \qquad (2.2)$$

Adding the above equations, we get:

$$\frac{\partial u(x,t)}{\partial t}+\sqrt{\frac{L}{C}}\frac{\partial i(x,t)}{\partial t}+\frac{1}{C}\frac{\partial i(x,t)}{\partial x}+\frac{1}{\sqrt{LC}}\frac{\partial u(x,t)}{\partial x}=0 \quad \Rightarrow$$
$$\frac{\partial}{\partial t}\left(u(x,t)+\sqrt{\frac{L}{C}}\ i(x,t)\right)+\frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\left(u(x,t)+\sqrt{\frac{L}{C}}\ i(x,t)\right)=0.$$

Next, we subtract the equations of (2.2):

$$\frac{\partial u(x,t)}{\partial t}-\sqrt{\frac{L}{C}}\frac{\partial i(x,t)}{\partial t}+\frac{1}{C}\frac{\partial i(x,t)}{\partial x}-\frac{1}{\sqrt{LC}}\frac{\partial u(x,t)}{\partial x}=0\Rightarrow$$
$$\frac{\partial}{\partial t}\left(u(x,t)-\sqrt{\frac{L}{C}}\,i(x,t)\right)-\frac{1}{\sqrt{LC}}\frac{\partial}{\partial x}\left(u(x,t)-\sqrt{\frac{L}{C}}\,i(x,t)\right)=0$$

and then, in view of the usually accepted denotations $Z_0=\sqrt{L/C}$, $v=1/\sqrt{LC}$, we get:

$$\frac{\partial}{\partial t}\big(u(x,t)+Z_0\,i(x,t)\big)+v\frac{\partial}{\partial x}\big(u(x,t)+Z_0\,i(x,t)\big)=0,$$
$$\frac{\partial}{\partial t}\big(u(x,t)-Z_0\,i(x,t)\big)-v\frac{\partial}{\partial x}\big(u(x,t)-Z_0\,i(x,t)\big)=0. \qquad (2.3)$$

Let us put

$$U(x,t)=u(x,t)+Z_0\,i(x,t),$$
$$I(x,t)=u(x,t)-Z_0\,i(x,t) \quad ,$$

that is $\begin{bmatrix}U(x,t)\\ I(x,t)\end{bmatrix}=\begin{bmatrix}1 & Z_0\\ 1 & -Z_0\end{bmatrix}\begin{bmatrix}u(x,t)\\ i(x,t)\end{bmatrix}$ and $\begin{bmatrix}u(x,t)\\ i(x,t)\end{bmatrix}=\begin{bmatrix}1/2 & 1/2\\ 1/(2Z_0) & -1/(2Z_0)\end{bmatrix}\begin{bmatrix}U(x,t)\\ I(x,t)\end{bmatrix}$ or

$$u(x,t)=\frac{1}{2}U(x,t)+\frac{1}{2}I(x,t),$$
$$i(x,t)=\frac{1}{2Z_0}U(x,t)-\frac{1}{2Z_0}I(x,t).$$

Then (2.3) becomes

$$\frac{\partial U(x,t)}{\partial t}+v\frac{\partial U(x,t)}{\partial x}=0,$$
$$\frac{\partial I(x,t)}{\partial t}-v\frac{\partial I(x,t)}{\partial x}=0.$$

The characteristics of the system are the families of straight lines

$x-vt=\text{const}$, $x+vt=\text{const}$.

For $x=0$ and $x=\Lambda$ we have

$$u(0,t)=\frac{U(0,t)}{2}+\frac{I(0,t)}{2},\ i(0,t)=-\frac{U(0,t)}{2Z_0}+\frac{I(0,t)}{2Z_0},$$
$$u(\Lambda,t)=\frac{U(\Lambda,t)}{2}+\frac{I(\Lambda,t)}{2},\ i(\Lambda,t)=-\frac{U(\Lambda,t)}{2Z_0}+\frac{I(\Lambda,t)}{2Z_0}.$$

Substituting in (1.4) and (1.5), we obtain

$$\begin{aligned}
&\frac{d\tilde{C}_0(u_{G_0C_0})}{du_{G_0C_0}}\frac{du_{G_0C_0}}{dt}=-\frac{U(0,t)}{2Z_0}+\frac{I(0,t)}{2Z_0}-G_0(u_{G_0C_0}),\\
&\frac{d\tilde{L}_0\left(-\frac{U(0,t)}{2Z_0}+\frac{I(0,t)}{2Z_0}\right)}{di_{G_0C_0}}\frac{d}{dt}\left(-\frac{U(0,t)}{2Z_0}+\frac{I(0,t)}{2Z_0}\right)=\frac{U(0,t)}{2}+\frac{I(0,t)}{2}-u_{G_0C_0}(t)+E_0(t),\\
&\frac{d\tilde{C}_1(u_{G_1C_1})}{du_{G_1C_1}}\frac{du_{G_1C_1}}{dt}=-\frac{U(\Lambda,t)}{2Z_0}+\frac{I(\Lambda,t)}{2Z_0}-G_1(u_{G_1C_1}),\\
&\frac{d\tilde{L}_1\left(-\frac{U(\Lambda,t)}{2Z_0}+\frac{I(\Lambda,t)}{2Z_0}\right)}{di_{G_1C_1}}\frac{d}{dt}\left(-\frac{U(\Lambda,t)}{2Z_0}+\frac{I(\Lambda,t)}{2Z_0}\right)=-\frac{U(\Lambda,t)}{2}-\frac{I(\Lambda,t)}{2}+u_{G_1C_1}(t)-E_1(t).
\end{aligned}\tag{2.4}$$

But $U(0,t)=U(\Lambda,t+T),\ I(0,t+T)=I(\Lambda,t)$.

It follows that $U(0,t-T)=U(\Lambda,t),\ I(0,t)=I(\Lambda,t-T)$.

We assume that the unknown functions are

$$U(0,t) \equiv U(t),\ I(\Lambda,t) \equiv I(t). \tag{2.5}$$

Consequently,

$$u(0,t) = \frac{U(0,t)+I(0,t)}{2} = \frac{U(t)+I(t-T)}{2},\ u(\Lambda,t) = \frac{U(\Lambda,t)+I(\Lambda,t)}{2} = \frac{U(t-T)+I(t)}{2},$$
$$i(0,t) = -\frac{U(0,t)}{2Z_0} + \frac{I(0,t)}{2Z_0} = -\frac{U(t)}{2Z_0} + \frac{I(t-T)}{2Z_0},\ \ i(\Lambda,t) = -\frac{U(\Lambda,t)}{2Z_0} + \frac{I(\Lambda,t)}{2Z_0} = -\frac{U(t-T)}{2Z_0} + \frac{I(t)}{2Z_0}.$$

Solving (2.4) with respect to the derivatives, we reach the system:

$$\frac{du_{G_0C_0}}{dt} = \frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}}\left(-\frac{U(0,t)}{2Z_0} + \frac{I(\Lambda,t-T)}{2Z_0} - G_0(u_{G_0C_0})\right),$$
$$\frac{dU(t)}{dt} = \frac{dI(t-T)}{dt} - \frac{2Z_0}{d\tilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{I(t-T)}{2Z_0}\right)/di_{G_0C_0}}\left(\frac{U(t)}{2}+\frac{I(t-T)}{2} - u_{G_0C_0}(t) + E_0(t)\right),$$
$$\frac{du_{G_1C_1}(t)}{dt} = \frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}}\left(-\frac{U(t-T)}{2Z_0} + \frac{I(t)}{2Z_0} - G_1(u_{G_1C_1})\right)$$
$$\frac{d}{dt}\left(\frac{-U(0,t-T)+I(\Lambda,t)}{2Z_0}\right) = \frac{1}{d\tilde{L}_1\left(-\frac{U(t-T)}{2Z_0}+\frac{I(t)}{2Z_0}\right)/di_{G_1C_1}}\left(-\frac{U(t-T)}{2} - \frac{I(t)}{2} + u_{G_1C_1}(t) - E_1(t)\right).$$

Then, in view of (2.5),

$$\frac{du_{G_0C_0}(t)}{dt} = \frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}}\left(-\frac{U(t)}{2Z_0} + \frac{I(t-T)}{2Z_0} - G_0(u_{G_0C_0})\right),$$
$$\frac{dU(t)}{dt} = \frac{dI(t-T)}{dt} - \frac{2Z_0}{d\tilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{I(t-T)}{2Z_0}\right)/di_{G_0C_0}}\left(\frac{U(t)}{2}+\frac{I(t-T)}{2} - u_{G_0C_0}(t) + E_0(t)\right),$$
$$\frac{du_{G_1C_1}(t)}{dt} = \frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}}\left(-\frac{U(t-T)}{2Z_0} + \frac{I(t)}{2Z_0} - G_1(u_{G_1C_1})\right),$$
$$\frac{dI(t)}{dt} = \frac{dU(t-T)}{dt} + \frac{2Z_0}{d\tilde{L}_1\left(-\frac{U(t-T)}{2Z_0}+\frac{I(t)}{2Z_0}\right)/di_{G_1C_1}}\left(-\frac{U(t-T)}{2} - \frac{I(t)}{2} + u_{G_1C_1}(t - E_1(t)\right)$$

Obviously, the system obtained is a neutral one with respect to the unknown functions $U(t)$ and $I(t)$. In order to formulate conditions for existence-uniqueness of the periodic solution, we translate the initial functions along the characteristics from $[0,\Lambda] \subset Ox$ to $[0,T] \subset Ot$.

The initial functions defined on $[0,T] \subset Ot$ are denoted by $U_0(t)$ and $I_0(t)$.

2.3 Analysis of the Arising Nonlinearities

First, we precise the definition domains of the functions:

$$\frac{d\tilde{C}_p(u)}{dt} = \frac{d(C_p(u).u)}{dt}, (p=0,1),$$

where $C_p(u) = \frac{c_p}{\sqrt[h]{(1-u/\Phi_p)}} = \frac{c_p\sqrt[h]{\Phi_p}}{\sqrt[h]{\Phi_p - u}}$, $c_p > 0,\ \Phi_p > 0,\ h \in [2,3]$ are constants and $|u| \le \phi_0 < \min\{\Phi_0, \Phi_1\}$. We have to show an interval for u where

$$\tilde{C}_p(u) = u.C_p(u) = c_p\sqrt[h]{\Phi_p}\frac{u}{\sqrt[h]{\Phi_p - u}}$$

has a strictly positive lower bound.

First, we calculate the derivatives

$$\frac{dC_p(u)}{du} = \frac{c_p\sqrt[h]{\Phi_p}}{h}(\Phi_p - u)^{-\frac{1+h}{h}}; \quad \frac{d^2C_p(u)}{du^2} = \frac{c_p\sqrt[h]{\Phi_p}}{h}\frac{1+h}{h}(\Phi_p - u)^{-\frac{1+2h}{h}};$$

$$\frac{d\tilde{C}_p(u)}{du} = C_p(u) + u\frac{dC_p(u)}{du} =$$

$$= c_p\sqrt[h]{\Phi_p}(\Phi_p - u)^{-\frac{1}{h}} + c_p\sqrt[h]{\Phi_p}\frac{u}{h}(\Phi_p - u)^{-\frac{1+h}{h}} = c_p\sqrt[h]{\Phi_p}(\Phi_p - u)^{-\frac{1}{h}-1}\left(\Phi_p - \left(1 - \frac{1}{h}\right)u\right);$$

$$\frac{d^2\tilde{C}_p(u)}{du^2} = 2\frac{dC_p(u)}{du} + u\frac{d^2C_p(u)}{du^2} = \frac{c_p\sqrt[h]{\Phi_p}}{h}(\Phi_p - u)^{-\frac{1+h}{h}} + u\frac{c_p\sqrt[h]{\Phi_p}}{h}\frac{1+h}{h}(\Phi_p - u)^{-\frac{1+2h}{h}} =$$

$$= \frac{2c_p\sqrt[h]{\Phi_p}}{h(\Phi_p - u)^{\frac{1+2h}{h}}}\left(\Phi_p + \frac{1}{h}u\right).$$

Since $|u| \le \phi_0 < \min\{\Phi_0, \Phi_1\} < \min\left\{\frac{h}{h-1}\Phi_0, \frac{h}{h-1}\Phi_1\right\}$, the derivative $\frac{d\tilde{C}_p(u)}{du} > 0$. Then

$$\min\{\tilde{C}_p(u) : u \in [-\phi_0.\phi_0]\} = \min\left\{c_p\sqrt[h]{\Phi_p}\frac{\phi_0}{\sqrt[h]{\Phi_p - \phi_0}}, c_p\sqrt[h]{\Phi_p}\left|\frac{-\phi_0}{\sqrt[h]{\Phi_p + \phi_0}}\right|\right\} = \frac{c_p\sqrt[h]{\Phi_p}\,\phi_0}{\sqrt[h]{\Phi_p + \phi_0}} = \hat{C}_p > 0.$$

Further on, we have

$$\left|\frac{d\tilde{C}_p(u)}{du}\right| \le \frac{2c_p\sqrt[h]{\Phi_p}}{h(\Phi_p - \phi_0)^{\frac{1}{h}+1}} + |u|\frac{(1+h)2c_p\sqrt[h]{\Phi_p}}{h^2(\Phi_p - \phi_0)^{\frac{1}{h}+2}} = \frac{2c_p\sqrt[h]{\Phi_p}\left[h(\Phi_p - \phi_0) + |u|(1+h)\right]}{h^2\ \sqrt[h]{(\Phi_p - \phi_0)^{1+2h}}} \le$$

$$\le \frac{2c_p \sqrt[h]{\Phi_p}\left[h\left(\Phi_p - \phi_0\right) + \phi_0(1+h)\right]}{h^2 \sqrt[h]{\left(\Phi_p - \phi_0\right)^{1+2h}}} = \frac{2c_p \sqrt[h]{\Phi_p}\left(h\Phi_p + \phi_0\right)}{h^2 \sqrt[h]{\left(\Phi_p - \phi_0\right)^{1+2h}}}.$$

For $-\phi_0 \le u \le \phi_0 < \min\left\{\frac{h}{h+1}, \Phi_0, \Phi_1\right\}$ it follows that $\frac{d\tilde{C}_p(u)}{du} = \frac{2c_p \sqrt[h]{\Phi_p}}{h\left(\Phi_p - u\right)^{\frac{1}{h}+2}} \frac{h-(1+h)u}{h} > 0$.

Therefore, $\min\left\{\tilde{C}_p(u) : u \in [-\phi_0, \phi_0]\right\} = \tilde{C}_p(-\phi_0) = \frac{c_p \sqrt[h]{\Phi_p}}{h} \frac{h\Phi_p + (h-1)\phi_0}{\left(\Phi_p + \phi_0\right)^{1+\frac{1}{h}}} = \hat{C}_p$.

For the $I-V$ characteristics, we assume $R_0(i) = \sum_{n=1}^{m} r_n^{(0)} i^n, (k=0,1)$ and $L_p(i) = \sum_{n=0}^{m} l_n^{(p)} i^n$. Then $\tilde{L}_p(i) = i \,. L_p(i) = i . \sum_{n=0}^{m} l_n^{(p)} i^n$.

For $\tilde{L}_p(i)$ we get $\frac{d\tilde{L}_p(i)}{di} = i\frac{dL_p(i)}{di} + L_p(i) = i\sum_{n=1}^{m} n l_n^{(p)} i^{n-1} + \sum_{n=1}^{m} l_n^{(p)} i^n = \sum_{n=1}^{m} (n+1) l_n^{(p)} i^n$.

Assumptions (L): $\left|i(t)\right| \le \mathrm{I}_0 < \infty \quad \Rightarrow \quad d\tilde{L}_p(i)(t)/di = \sum_{n=1}^{m} (n+1) l_n^{(p)} \left(i(t)\right)^n \ge \hat{L}_p > 0;$

Assumptions (C): $\left|u(0,t)\right| \le \frac{e^{\mu T_0}\left(V_0 + J_0\right)}{2} \le \phi_0;\ \left|u(\Lambda,t)\right| \le \frac{e^{\mu T_0}\left(V_0 + J_0\right)}{2} \le \phi_0;$

$$\left|i(0,t)\right| \le \frac{e^{\mu T_0}\left(V_0 + J_0\right)}{2Z_0} \le \mathrm{I}_0;\quad \left|i(\Lambda,t)\right| \le \frac{e^{\mu T_0}\left(V_0 + J_0\right)}{2Z_0} \le \mathrm{I}_0;$$

Assumptions (G): $G_0(i_{R_0 L_0}) = \sum_{n=1}^{m} g_n^{(0)} (i_{R_0 L_0})^n$, $G_1(i_{R_1 L_1}) = \sum_{n=1}^{m} g_n^{(1)} (i_{R_1 L_1})^n$.

We introduce the sets:

$$\begin{aligned}
M_0 &= \left\{ u_{G_0 C_0}(t) \in C_{T_0}^1[T,\infty) : \left|u_{G_0 C_0}(t)\right| \le U_{G_0} e^{\mu(t-T-kT_0)}, t \in [T+kT_0, T+(k+1)T_0] \right\}, \\
M_U &= \left\{ U(.) \in C_{T_0}^1[T,\infty) : \left|U(t)\right| \le V_0 e^{\mu(t-T-kT_0)}, t \in [T+kT_0, T+(k+1)T_0] \right\}, \\
M_1 &= \left\{ u_{G_1 C_1}(t) \in C_{T_0}^1[T,\infty) : \left|u_{G_1 C_1}(t)\right| \le U_{G_1} e^{\mu(t-T-kT_0)}, t \in [T+kT_0, T+(k+1)T_0] \right\}, \\
M_I &= \left\{ I(.) \in C_{T_0}^1[T,\infty) : \left|I(t)\right| \le J_0 e^{\mu(t-T-kT_0)}, t \in [T+kT_0, T+(k+1)T_0] \right\},
\end{aligned}$$

where $I_{G_0}, V_0, I_{G_1}, T_0, \mu$ are positive constants (chosen below). We introduce the constant $\mu_0 = \mu T_0$.

The set $M_0 \times M_U \times M_1 \times M_I$ turns out into a complete metric space (for details [20]) with respect to the metric

$$\rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})) =$$
$$= \max\Big\{ \hat{\rho}(u_{G_0C_0}, \bar{u}_{G_0C_0}),\ \rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0}),\ \hat{\rho}(U, \bar{U}),\ \rho_\mu^{(k)}(\dot{U}, \dot{\bar{U}}),\ \hat{\rho}(u_{G_1C_1}, \bar{u}_{G_1C_1}),\ \rho_\mu^{(k)}(\dot{u}_{G_1C_1}, \dot{\bar{u}}_{G_1C_1}),$$
$$\hat{\rho}(I, \bar{I}),\ \rho_\mu^{(k)}(\dot{I}, \dot{\bar{I}}):\ k = 0,1,\ldots,m-1 \Big\}$$

where

$$\rho^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) = \max\left\{\left|u_{G_0C_0}(t) - \bar{u}_{G_0C_0}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho^{(k)}(U, \bar{U}) = \max\left\{\left|U(t) - \bar{U}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho^{(k)}(u_{G_1C_1}, \bar{u}_{G_1C_1}) = \max\left\{\left|u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho^{(k)}(I, \bar{I}) = \max\left\{\left|I(t) - \bar{I}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\hat{\rho}(u_{G_0C_0}, \bar{u}_{G_0C_0}) = \max\left\{\left|u_{G_0C_0}(t) - \bar{u}_{G_0C_0}(t)\right| : t \in [T, 2T]\right\},$$
$$\hat{\rho}(U, \bar{U}) = \max\left\{\left|U(t) - \bar{U}(t)\right| : t \in [T, 2T]\right\},$$
$$\hat{\rho}(u_{G_1C_1}, \bar{u}_{G_1C_1}) = \max\left\{\left|u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t)\right| : t \in [T, 2T]\right\},\ \hat{\rho}(I, \bar{I}) = \max\left\{\left|I(t) - \bar{I}(t)\right| : t \in [T, 2T]\right\},$$
$$\rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|u_{G_0C_0}(t) - \bar{u}_{G_0C_0}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|\dot{u}_{G_0C_0}(t) - \dot{\bar{u}}_{G_0C_0}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho_\mu^{(k)}(U, \bar{U}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|U(t) - \bar{U}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho_\mu^{(k)}(\dot{U}, \dot{\bar{U}}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|\dot{U}(t) - \dot{\bar{U}}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho_\mu^{(k)}(u_{G_1C_1}, \bar{u}_{G_1C_1}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho_\mu^{(k)}(\dot{u}_{G_1C_1}, \bar{\bar{u}}_{G_1C_1}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|\dot{u}_{G_1C_1}(t) - \dot{\bar{u}}_{G_1C_1}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho_\mu^{(k)}(I, \bar{I}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|I(t) - \bar{I}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\},$$
$$\rho_\mu^{(k)}(\dot{I}, \dot{\bar{I}}) = \max\left\{e^{-\mu(t-T-kT_0)}\left|\dot{I}(t) - \dot{\bar{I}}(t)\right| : t \in [T + kT_0, T + (k+1)T_0]\right\}.$$

Remark 3.1. It is easy to verify that

$$\hat{\rho}(u_{G_0C_0}, \bar{u}_{G_0C_0}) = \max\left\{\rho^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) : k = 0,1,2,\ldots,m-1\right\},$$
$$\hat{\rho}(U, \bar{U}) = \max\left\{\rho^{(k)}(U, \bar{U}) : k = 0,1,2,\ldots,m-1\right\},$$
$$\hat{\rho}(u_{G_1C_1}, \bar{u}_{G_1C_1}) = \max\left\{\rho^{(k)}(u_{G_1C_1}, \bar{u}_{G_1C_1}) : k = 0,1,2,\ldots,m-1\right\},$$
$$\hat{\rho}(I, \bar{I}) = \max\left\{\rho^{(k)}(I, \bar{I}) : k = 0,1,2,\ldots,m-1\right\},\ \rho^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) \le e^{\mu_0}\rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}),$$

$$\rho^{(k)}(U,\overline{U}) \le e^{\mu_0}\rho_\mu^{(k)}(U,\overline{U}),\ \ \rho^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1}) \le e^{\mu_0}\rho_\mu^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1}),$$
$$\rho^{(k)}(I,\bar{I}) \le e^{\mu_0}\rho_\mu^{(k)}(I,\bar{I}),$$
$$\hat{\rho}(u_{G_0C_0},\overline{u}_{G_0C_0}) = \max\left\{\rho^{(k)}(u_{G_0C_0},\overline{u}_{G_0C_0}) : k = 0,1,2,...,m-1\right\} \le$$
$$\le e^{\mu_0}\max\left\{\rho_\mu^{(k)}(u_{G_0C_0},\overline{u}_{G_0C_0}) : k = 0,1,2,...,2m-1\right\} \le e^{\mu_0}\rho_\mu\left((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\bar{I})\right),$$
$$\hat{\rho}(U,\overline{U}) = \max\left\{\rho^{(k)}(U,\overline{U}) : k = 0,1,2,...,m-1\right\} \le$$
$$\le e^{\mu_0}\max\left\{\rho_\mu^{(k)}(U,\overline{U}) : k = 0,1,2,...,2m-1\right\} \le e^{\mu_0}\rho_\mu\left((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\bar{I})\right),$$
$$\hat{\rho}(u_{G_1C_1},\overline{u}_{G_1C_1}) = \max\left\{\rho^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1})) : k = 0,1,2,...,m-1\right\} \le$$
$$\le e^{\mu_0}\max\left\{\rho_\mu^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1}) : k = 0,1,2,...,2m-1\right\} \le e^{\mu_0}\rho_\mu\left((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\bar{I})\right),$$
$$\hat{\rho}(I,\bar{I}) = \max\left\{\rho^{(k)}(I,\bar{I}) : k = 0,1,2,...,m-1\right\} \le$$
$$\le e^{\mu_0}\max\left\{\rho_\mu^{(k)}(I,\bar{I}) : k = 0,1,2,...,2m-1\right\} \le e^{\mu_0}\rho_\mu\left((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\bar{I})\right).$$

2.4 Operator Presentation of the Periodic Problem

Now we formulate the main problem: to find a T_0-periodic solution $\left(u_{G_0C_0}(t), U(t), u_{G_1C_1}(t), I(t)\right)$ of the system:

$$\frac{du_{G_0C_0}(t)}{dt} = \frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}}\left(-\frac{U(t)}{2Z_0} + \frac{I(t-T)}{2Z_0} - G_0(u_{G_0C_0})\right),$$
$$\frac{dU(t)}{dt} = \frac{dI(t-T)}{dt} - \frac{2Z_0}{d\tilde{L}_0\left(-\frac{U(t)}{2Z_0} + \frac{I(t-T)}{2Z_0}\right)/di_{G_0C_0}}\left(\frac{U(t)}{2} + \frac{I(t-T)}{2} - u_{G_0C_0}(t) + E_0(t)\right),$$
$$\frac{du_{G_1C_1}(t)}{dt} = \frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}}\left(-\frac{U(t-T)}{2Z_0} + \frac{I(t)}{2Z_0} - G_1(u_{G_1C_1})\right), \tag{4.1}$$
$$\frac{dI(t)}{dt} = \frac{dU(t-T)}{dt} + \frac{2Z_0}{d\tilde{L}_1\left(-\frac{U(t-T)}{2Z_0} + \frac{I(t)}{2Z_0}\right)/di_{G_1C_1}}\left(\frac{U(t-T)}{2} + \frac{I(t)}{2} - u_{G_1C_1}(t) + E_1(t)\right),$$
$$u_{G_0C_0}(T) = u_{G_0C_0}^{(0)},\ U(t) = U^{(0)}(t),\ t \in [0,T];\ \ u_{G_1C_1}(T) = u_{G_1C_1}^{(0)};\ I(t) = I^{(0)}(t),\ t \in [0,T].$$

By $C_{T_0}^1[T,\infty)$ we mean the space of all continuous T_0-periodic functions with piece wise continuous derivatives. The main difficulty is to define a suitable operator whose fixed points are the solutions sought. The operator $B = (B_0, B_U, B_1, B_I)$ is defined for any $t \in [T + kT_0, T + (k+1)T_0]$, $(k = 0,1,2,...,m-1)$ by the formulas (assuming that $u_{G_0C_0}^{(0)}(T) = 0$, $U^{(0)}(T) = 0, u_{G_1C_1}^{(0)} = 0, I^{(0)}(T) = 0$):

$$B_0^{(k)}(u_{G_0C_0},U,I)(t):=\int_{T+kT_0}^{t}U_{G_0}(s)ds-\frac{t-T-kT_0}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}U_{G_0}(s)ds,$$
$$B_U^{(k)}(u_{G_0C_0},U,I)(t):=\int_{T+kT_0}^{t}U(s)ds-\frac{t-T-kT_0}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}U(s)ds,$$
$$B_1^{(k)}(U,I,u_{G_1C_1})(t):=\int_{T+kT_0}^{t}U_{G_1}(s)ds-\frac{t-T-kT_0}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}U_{G_1}(s)ds, \quad (4.2)$$
$$B_I^{(k)}(U,I,u_{G_1C_1})(t):=\int_{T+kT_0}^{t}I(s)ds-\frac{t-T-kT_0}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}I(s)ds,$$

$t\in[T+kT_0,T+(k+1)T_0]$ $(k=0,1,2,...,m-1)$, where the functions $U(t-T)$ and $I(t-T)$ are replaced by the initial functions on the segment $[T,2T]$ translated to the right, that is

$$U(t-T)=\vec{U}_0(t),\ I(t-T)=\vec{I}_0(t),\ t\in[T,2T]:$$
$$U_{G_0}(u_{G_0C_0},U,\vec{I}_0)(t)\equiv\frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}}\left(-\frac{U(t)}{2Z_0}+\frac{\vec{I}_0(t)}{2Z_0}-G_0(u_{G_0C_0})\right),$$
$$U(u_{G_0C_0},U,\vec{I}_0)(t)\equiv\frac{d\vec{I}_0(t)}{dt}-\frac{2Z_0}{d\widetilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{\vec{I}_0(t)}{2Z_0}\right)/di_{G_0C_0}}\left(\frac{U(t)}{2}+\frac{\vec{I}_0(t)}{2}-u_{G_0C_0}(t)+E_0(t)\right),$$
$$U_{G_1}(\vec{U}_0,u_{G_1C_1},I)(t)\equiv\frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}}\left(-\frac{\vec{U}_0(t)}{2Z_0}+\frac{I(t)}{2Z_0}-G_1(u_{G_1C_1})\right),$$
$$I(\vec{U}_0,u_{G_1C_1},I)\equiv\frac{d\vec{U}_0(t)}{dt}+\frac{2Z_0}{d\widetilde{L}_1\left(-\frac{\vec{U}_0(t)}{2Z_0}+\frac{I(t)}{2Z_0}\right)/di_{G_1C_1}}\left(\frac{\vec{U}_0(t)}{2}+\frac{I(t)}{2}-u_{G_1C_1}(t)+E_1(t)\right).$$

From now on, the following assumptions will be fulfilled:

(E): $E_p(.)\in C^1_{T_0}[0,\infty),(p=0,1);\ \left|E_p(t)\right|\le U_{E_p}e^{\mu(t-T-kT_0)}\le V_0e^{\mu(t-T-kT_0)};$

(IN): $U_0(.),I_0(.)\in C^1_{T_0}[0,T],\ T=mT_0,\ \left|U_0(t)\right|\le e^{-\beta}V_0e^{\mu(t-kT_0)},\ \left|I_0(t)\right|\le e^{-\beta}J_0e^{\mu(t-kT_0)}$
$(k=0,1,2,...,m-1)$ where $\beta=\text{const.}>0$ is chosen below.
It follows that $\left|\vec{U}_0(t)\right|\le e^{-\beta}V_0e^{\mu(t-T-kT_0)},\ \left|\vec{I}_0(t)\right|\le e^{-\beta}J_0e^{\mu(t-T-kT_0)}(k=0,1,2,...,m-1)$.

Lemma 4.1. If **(E)** and **(IN)** are satisfied and $(u_{G_0C_0},U,u_{G_1C_1},I)\in M_0\times M_U\times M_1\times M_I$, then operator functions $U_{G_0}(i_{R_0L_0},U,I)(t)$, $U(u_{G_0C_0},U,I)(t)$, $U_{G_1}(U,u_{G_1C_1},I)(t)$, $I(U,u_{G_1C_1},I)$ are T_0-periodic ones.
The proof is omitted.

Lemma 4.2. If $(u_{G_0C_0}, U, u_{G_1C_1}, I) \in M_0 \times M_U \times M_1 \times M_I$, then

$$\left(B_0(u_{G_0C_0}, U, I)(t), B_U(u_{G_0C_0}, U, I)(t), B_1(U, I, u_{G_1C_1})(t), B_I^{(k)}(U, I, u_{G_1C_1})(t)\right) \in M_0 \times M_U \times M_1 \times M_I.$$

Proof: We notice that

$$B_0(u_{G_0C_0}, U, I)(t),\ B_U(u_{G_0C_0}, U, I)(t),\ B_1(U, I, u_{G_1C_1})(t),\ B_I^{(k)}(U, I, u_{G_1C_1})(t)$$

are continuously differentiable functions. Indeed,

$$B_0^{(k)}(u_{G_0C_0}, U, I)(T + kT_0) := \int_{T+kT_0}^{T+kT_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds - \frac{T + kT_0 - T - kT_0}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds = 0,$$

$$B_0^{(k)}(u_{G_0C_0}, U, I)(T + (k+1)T_0) := \int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s) -$$

$$- \frac{T + (k+1)T_0 - T - kT_0}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds = 0,$$

$$B_0^{(k)} U_{G_0}(i_{R_0L_0}, U, I)(T + (k+1)T_0) := \int_{T+(k+1)T_0}^{T+(k+1)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds -$$

$$- \frac{T + (k+1)T_0 - T - (k+1)T_0}{T_0} \int_{T+(k+1)T_0}^{T+(k+2)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds = 0.$$

For the derivatives we obtain

$$\frac{dB_0^{(k)}(u_{G_0C_0}, U, I)(t)}{dt} = U_{G_0}(i_{R_0L_0}, U, I)(t) - \frac{1}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds,$$

$$\frac{dB_0^{(k+1)}(u_{G_0C_0}, U, I)(t)}{dt} = U_{G_0}(i_{R_0L_0}, U, I)(t) - \frac{1}{T_0} \int_{T+(k+1)T_0}^{T+(k+2)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds$$

and therefore

$$\frac{dB_0^{(k)}(u_{G_0C_0}, U, I)(T + (k+1)T_0)}{dt} = U_{G_0}(i_{R_0L_0}, U, I)(T + (k+1)T_0) - \frac{1}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds =$$

$$= U_{G_0}(i_{R_0L_0}, U, I)(T + (k+1)T_0) - \frac{1}{T_0} \int_{T+(k+1)T_0}^{T+(k+2)T_0} U_{G_0}(i_{R_0L_0}, U, I)(s)ds = \frac{dB_0^{(k+1)}(u_{G_0C_0}, U, I)(T + (k+1)T_0)}{dt}.$$

Analogous reasoning’s might be applied to the operator functions

$B_U(u_{G_0C_0}, U, I)(t), B_1(U, u_{G_1C_1}, I)(t), B_I^{(k)}(U, u_{G_1C_1}, I)(t)$.

It is easy to verify that if $(u_{G_0C_0}(t), U(t), u_{G_1C_1}(t), I(t)$ are T_0 -periodic functions, then

$B_0(u_{G_0C_0}, U, I)(t), B_U(u_{G_0C_0}, U, I)(t), B_1(U, u_{G_1C_1}, I)(t), B_I^{(k)}(U, u_{G_1C_1}, I)(t)$

are T_0 -periodic too (on $[T, \infty)$).

Lemma 4.2 is thus proved.

Remark 4.1. We use the inequality

$$\frac{e^{n\mu T_0}-1}{n} = \frac{(e^{\mu T_0}-1)(e^{(n-1)\mu T_0}+e^{(n-2)\mu T_0}+...+1)}{n} \leq \frac{(e^{\mu T_0}-1)ne^{(n-1)\mu T_0}}{n} = (e^{\mu T_0}-1)e^{(n-1)\mu T_0}.$$

Lemma 4.3. The periodic problem (4.1) has a solution

$(u_{G_0C_0}, U, u_{G_1C_1}, I) \in M_0 \times M_U \times M_1 \times M_I$

iff the operator *B* has a fixed point $(u_{G_0C_0}, U, u_{G_1C_1}, I) \in M_0 \times M_U \times M_1 \times M_I$, that is,

$$u_{G_0C_0} = B_0(u_{G_0C_0}, U, I),\ U = B_U(u_{G_0C_0}, U, I)(t),\ u_{G_1C_1} = B_1(U, u_{G_1C_1}, I),\ I = B_I^{(k)}(U, u_{G_1C_1}, I)(t).$$

***Proof*:** Let $(u_{G_0C_0}, U, u_{G_1C_1}, I) \in M_0 \times M_U \times M_1 \times M_I$ be a T_0 -periodic solution of (4.1). Then, after integration of the first equation, we have ($u_{G_0C_0}(T)=0$):

$$u_{G_0C_0}(t) = \int_{T+kT_0}^{t} U_{G_0}(t)dt \Rightarrow u_{G_0C_0}(T+(k+1)T_0) = \int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(t)dt \Rightarrow \int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(t)dt = 0.$$

Therefore, $B_0^{(k)}(u_{G_0C_0}, U, I)(t) = \int_{T+kT_0}^{t} U_{G_0}(s)ds$. Analogously we obtain

$$B_U^{(k)}(u_{G_0C_0}, U, I)(t) = \int_{T+kT_0}^{t} U(s)ds,\ B_1^{(k)}(U, u_{G_1C_1}, I) = \int_{T+kT_0}^{t} U_{G_1}(s)ds,\ B_I^{(k)}(U, u_{G_1C_1}, I) = \int_{T+kT_0}^{t} I(s)ds,$$

that is, $(u_{G_0C_0}, U, u_{G_1C_1}, I) \in M_0 \times M_U \times M_1 \times M_I$ is a fixed point of *B*.

Conversely, let *B* have a fixed point $(u_{G_0C_0}, U, u_{G_1C_1}, I) \in M_0 \times M_U \times M_1 \times M_I$. Then

$$\left|\int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}(u_{G_0C_0}, U, \bar{I}_0)(s)ds\right| \le \frac{1}{\hat{C}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{|U(s)|}{2Z_0}+\frac{|\bar{I}_0(s)|}{2Z_0}+\left|G_0(u_{G_0C_0}(s))\right|\right)ds \le$$

$$\le \frac{1}{\hat{C}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{V_0 e^{\mu(s-T-kT_0)}}{2Z_0}+\frac{e^{-\beta}J_0 e^{\mu(s-T-kT_0)}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|\left|u_{G_0C_0}(s)\right|^n\right)ds \le$$

$$\le \frac{1}{\hat{C}_0}\left(\frac{V_0}{2Z_0}\int_{T+kT_0}^{T+(k+1)T_0} e^{\mu(s-T-kT_0)}ds+\frac{e^{-\beta}J_0}{2Z_0}\int_{T+kT_0}^{T+(k+1)T_0} e^{\mu(s-T-kT_0)}ds+\sum_{n=1}^{m}\left|g_n^{(0)}\right|\int_{T+kT_0}^{T+(k+1)T_0} U_{G_0}^n e^{n\mu(s-T-kT_0)}\right) \le$$

$$\le \frac{V_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{e^{-\beta}J_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{1}{\hat{C}_0}\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n\frac{e^{n\mu T_0}-1}{n\mu} \le$$

$$\le \frac{V_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{e^{-\beta}J_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{1}{\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu T_0} \le$$

$$\le \frac{e^{\mu_0}-1}{\mu\hat{C}_0}\left(\frac{V_0+e^{-\beta}J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right) \equiv M_{U_{G_0}}(\mu);$$

$$\left|\int_{T+kT_0}^{T+(k+1)T_0} U(u_{G_0C_0}, U, \bar{I}_0)(s)ds\right| \le \left|\int_{T+kT_0}^{T+(k+1)T_0}\frac{d\bar{I}_0(s)}{ds}ds\right|+\frac{2Z_0}{\hat{L}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{|U(s)|}{2}+\frac{|\bar{I}_0(s)|}{2}+\left|u_{G_0C_0}(s)\right|+\left|E_0(s)\right|\right)ds \le$$

$$\le \frac{2Z_0}{\hat{L}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{V_0}{2}+\frac{J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)e^{\mu(s-T-kT_0)}ds \le \frac{Z_0\left(V_0+J_0e^{-\beta}+2U_{G_0}+2U_{E_0}\right)}{\hat{L}_0}\frac{e^{\mu T_0}-1}{\mu} \equiv M_U(\mu);$$

$$\left|\int_{T+kT_0}^{T+(k+1)T_0} U_{G_1}(\bar{U}_0, u_{G_1C_1}, I)(s)ds\right| \le \frac{1}{\hat{C}_1}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{\bar{U}_0(s)}{2Z_0}+\frac{I(s)}{2Z_0}+G_1(u_{G_1C_1}(s))\right)ds \le$$

$$\le \frac{e^{\mu_0}-1}{\mu\hat{C}_1}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|U_{G_1}^n e^{(n-1)\mu_0}\right) \equiv M_{U_{G_1}}(\mu);$$

$$\left|\int_{T+kT_0}^{T+(k+1)T_0} I(\bar{U}_0, u_{G_1C_1}, I)(s)ds\right| \le \left|\int_{T+kT_0}^{T+(k+1)T_0}\frac{d\bar{U}_0(s)}{ds}ds\right|+\frac{2Z_0}{\hat{L}_1}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{|\bar{U}_0(s)|}{2}+\frac{|I(s)|}{2}+\left|u_{G_1C_1}(s)\right|+\left|E_1(s)\right|\right)ds \le$$

$$\le \frac{Z_0\left(V_0e^{-\beta}+J_0+2U_{G_1}+2U_{E_1}\right)}{\hat{L}_1}\frac{e^{\mu T_0}-1}{\mu} \equiv M_I(\mu).$$

If we assume that $\left|\int_{T+kT_0}^{T+(k+1)T_0} U_{G_0C_0}(s)ds\right| \ne 0$ in view of $\lim_{\mu\to\infty} M_{U_{G_0}}(\mu)=0$, we obtain a contradiction. In the same way, we conclude that

$$\left|\int_{T+kT_0}^{T+(k+1)T_0} U(s)ds\right|=0\ ,\quad \int_{T+kT_0}^{T+(k+1)T_0}\left|U_{G_1}(t)\right|dt=0\ ,\quad \left|\int_{T+kT_0}^{T+(k+1)T_0} I(s)ds\right|=0\ .$$

Consequently, the components of operator B become:

$$u_{G_0C_0}(t)=\int_{T+kT_0}^{t}U_{G_0}(i_{R_0L_0},U,\vec{I}_0)(s)dt,\quad U(t)=\int_{T+kT_0}^{t}U(u_{G_0C_0},U,\vec{I}_0)(s)ds.$$

$$u_{G_1C_1}(t)=\int_{T+kT_0}^{t}U_{G_1}(\vec{U}_0,u_{G_1C_1},I)(s)ds,\quad I(t)=\int_{T+kT_0}^{t}I(\vec{U}_0,u_{G_1C_1},I)(s)ds.$$

Differentiating the last equalities, we conclude that (4.1) has a T_0-periodic solution. Lemma 4.3 is thus proved.

2.5 Lipschitz Estimates for the Right Hand Sides of the Operator

First, we notice that

$$\frac{\partial U_{G_0}(u_{G_0C_0},U,\vec{I}_0)}{\partial u_{G_0C_0}}=\frac{\partial}{\partial u_{G_0C_0}}\left[\frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}}\left(\frac{-U(t)+\vec{I}_0(t)}{2Z_0}-G_0(u_{G_0C_0})\right)\right]=$$

$$=-\frac{1}{\left(d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}\right)^2}\frac{d^2\overline{C}_0(u_{G_0C_0})}{du^2_{G_0C_0}}\left(\frac{-U(t)+\vec{I}_0(t)}{2Z_0}-G_0(u_{G_0C_0})\right)+\frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}}\left(-\frac{dG_0(u_{G_0C_0})}{du_{G_0C_0})}\right);$$

$$\frac{\partial U_{G_0}(u_{G_0C_0},U,\vec{I}_0)}{\partial U}=\frac{\partial}{\partial U}\left[\frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}}\left(\frac{-U(t)+\vec{I}_0(t)}{2Z_0}-G_0(u_{G_0C_0})\right)\right]=-\frac{1}{2Z_0}\frac{1}{d\overline{C}_0(u_{G_0C_0})/du_{G_0C_0}};$$

$$\frac{\partial U(u_{G_0C_0},U,\vec{I}_0)}{\partial u_{G_0C_0}}=\frac{\partial}{\partial u_{G_0C_0}}\left[\frac{d\vec{I}_0(t)}{dt}-\frac{2Z_0}{d\tilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{\vec{I}_0(t)}{2Z_0}\right)/di_{G_0C_0}}\left(\frac{U(t)}{2}+\frac{\vec{I}_0(t)}{2}-u_{G_0C_0}(t)+E_0(t)\right)\right]=$$

$$=\frac{2Z_0}{d\tilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{\vec{I}_0(t)}{2Z_0}\right)/di_{G_0C_0}};$$

$$\frac{\partial U(u_{G_0C_0},U,\vec{I}_0)}{\partial U}=\frac{\partial}{\partial U}\left[\frac{d\vec{I}_0(t)}{dt}-\frac{2Z_0}{d\tilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{\vec{I}_0(t)}{2Z_0}\right)/di_{G_0C_0}}\left(\frac{U(t)}{2}+\frac{\vec{I}_0(t)}{2}-u_{G_0C_0}(t)+E_0(t)\right)\right]=$$

$$=-\frac{1}{\left(d\tilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{\vec{I}_0(t)}{2Z_0}\right)/di_{G_0C_0}\right)^2}\frac{d\tilde{L}^2{}_0\left(-\frac{U(t)}{2Z_0}+\frac{\vec{I}_0(t)}{2Z_0}\right)}{di^2_{G_0C_0}}\left(\frac{U(t)}{2}+\frac{\vec{I}_0(t)}{2}-u_{G_0C_0}(t)+E_0(t)\right)+$$

$$+\frac{Z_0}{d\tilde{L}_0\left(-\frac{U(t)}{2Z_0}+\frac{\bar{I}_0(t)}{2Z_0}\right)/di_{G_0C_0}};$$

$$\frac{\partial U_{G_1}(\vec{U}_0,u_{G_1C_1},I)}{\partial u_{G_1C_1}}=\frac{\partial}{\partial u_{G_1C_1}}\left[\frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}}\left(\frac{-\vec{U}_0(t)+I(t)}{2Z_0}-G_1(u_{G_1C_1})\right)\right]=$$
$$=-\frac{1}{\left(d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}\right)^2}\frac{d^2\overline{C}_1(u_{G_1C_1})}{du^2_{G_1C_1}}\left(\frac{-\vec{U}_0(t)+I(t)}{2Z_0}-G_1(u_{G_1C_1})\right)+$$
$$+\frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}}\left(-\frac{dG_0(u_{G_1C_1})}{du_{G_1C_1}}\right);$$

$$\frac{\partial U_{G_1}(\vec{U}_0,u_{G_1C_1},I)}{\partial I}=\frac{\partial}{\partial I}\left[\frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}}\left(-\frac{\vec{U}_0(t)}{2Z_0}+\frac{I(t)}{2Z_0}-G_1(u_{G_1C_1})\right)\right]=\frac{1}{2Z_0}\frac{1}{d\overline{C}_1(u_{G_1C_1})/du_{G_1C_1}};$$

$$\frac{\partial I(\vec{U}_0,u_{G_1C_1},I)}{\partial u_{G_1C_1}}=-\frac{2Z_0}{d\tilde{L}_1\left(-\frac{\vec{U}_0(t)}{2Z_0}+\frac{I(t)}{2Z_0}\right)/di_{G_1C_1}};$$

$$\frac{\partial I(\vec{U}_0,u_{G_1C_1},I)}{\partial I}=\frac{1}{\left(d\tilde{L}_1\left(-\frac{\vec{U}_0(t)}{2Z_0}+\frac{I(t)}{2Z_0}\right)/di_{G_1C_1}\right)^2}\frac{d^2\tilde{L}_1\left(-\frac{\vec{U}_0(t)}{2Z_0}+\frac{I(t)}{2Z_0}\right)}{di^2{}_{G_1C_1}}\left(\frac{\vec{U}_0(t)}{2}+\frac{I(t)}{2}-u_{G_1C_1}(t)+E_1(t)\right)+$$
$$+\frac{Z_0}{d\tilde{L}_1\left(-\frac{\vec{U}_0(t)}{2Z_0}+\frac{I(t)}{2Z_0}\right)/di_{G_1C_1}}.$$

Since

$$\frac{1}{\left(d\tilde{C}_p(u_{G_pC_p})/du_{G_pC_p}\right)^2}\le\left(\frac{\sqrt[h]{\Phi_p+\phi_0}}{c_p\sqrt[h]{\Phi_p}\,\phi_0}\right)^2\equiv A_p{}^2,\quad\left|\frac{d^2\tilde{C}_p(u_{G_pC_p})}{du^2_{G_pC_p}}\right|\le\frac{2c_p\sqrt[h]{\Phi_p}\left(h\Phi_p+\phi_0\right)}{h^2\sqrt[h]{\left(\Phi_p-\phi_0\right)^{1+2h}}}\equiv B_p,$$

$$|i(t)|\le \mathrm{I}_0<\infty\quad\Rightarrow\quad d\tilde{L}_p(i)(t)/di=\sum_{n=1}^{m}(n+1)l_n^{(p)}\left(i(t)\right)^n\ge\hat{L}_p>0,\ (p=0,1)$$

we have

$$\left|\frac{\partial U_{G_0}(u_{G_0C_0},U,\bar{I}_0)}{\partial u_{G_0C_0}}\right| \le A_0{}^2B_0\left(e^{\mu_0}\,\frac{V_0+J_0e^{-\beta}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|\left(U_{G_0}\right)^n e^{n\mu_0}\right)+A_0\sum_{n=1}^{m}n\left|g_n^{(0)}\right|\left(U_{G_0}\right)^{n-1}e^{(n-1)\mu_0};$$

$$\left|\frac{\partial U_{G_0}(u_{G_0C_0},U,\bar{I}_0)}{\partial U}\right| \le \frac{1}{2Z_0}A_0;\quad \left|\frac{\partial U(u_{G_0C_0},U,\bar{I}_0)}{\partial u_{G_0C_0}}\right| \le \frac{2Z_0}{\hat{L}_0};$$

$$\left|\frac{\partial U(u_{G_0C_0},U,\bar{I}_0)}{\partial U}\right| \le \frac{1}{\left(\hat{L}_0\right)^2}\sum_{n=1}^{m}(n+1)nl_n^{(0)}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0+J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)+\frac{Z_0}{\hat{L}_0};$$

$$\left|\frac{\partial U_{G_1}(\bar{U}_0,u_{G_1C_1},I)}{\partial u_{G_1C_1}}\right| \le$$

$$A_1{}^2B_1\left(e^{\mu_0}\,\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0};$$

$$\left|\frac{\partial U_{G_1}(\bar{U}_0,u_{G_1C_1},I)}{\partial u_{G_1C_1}}\right| \le \frac{1}{2Z_0}A_1;\quad \left|\frac{\partial I(\bar{U}_0,u_{G_1C_1},I)}{\partial u_{G_1C_1}}\right| \le \frac{2Z_0}{\hat{L}_1};$$

$$\left|\frac{\partial I((\bar{U}_0,u_{G_1C_1},I))}{\partial I}\right| \le \frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}.$$

2.6 Existence-Uniqueness of a Periodic Solution

Here we formulate the main result concerning an existence-uniqueness of the periodic solution to the obtained neutral system.

Theorem 6.1. Let the following conditions **(L), (E), (IN), (C)** and **(G)** be fulfilled.
Then there exists a unique T_0-periodic solution of (5.1).

Proof: We show that B maps $M_0\times M_U\times M_1\times M_I$ into itself. Indeed, in view of

$$\left|\frac{t-T-kT_0}{T_0}\right|\le 1,\, t\in[T+kT_0,T+(k+1)T_0],$$

we have

$$\left|B_0^{(k)}(u_{G_0C_0},U,I)(t)\right|\le\left|\int_{T+kT_0}^{t}U_{G_0}(s)ds\right|+\left|\int_{T+kT_0}^{T+(k+1)T_0}U_{G_0}(s)ds\right|\equiv J_1+J_2.$$

But

$$J_1\le\frac{1}{\hat{C}_0}\int_{T+kT_0}^{t}\left(\frac{|U(s)|}{2Z_0}+\frac{\left|\bar{I}_0(s)\right|}{2Z_0}+\left|G_0(u_{G_0C_0}(s))\right|\right)ds\le$$

$$\le \frac{1}{\hat{C}_0}\int_{T+kT_0}^{t}\left(\frac{V_0 e^{\mu(s-T-kT_0)}}{2Z_0}+\frac{e^{-\beta}J_0 e^{\mu(s-T-kT_0)}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|\left|u_{G_0C_0}(s)\right|^n\right)ds \le$$

$$\le \frac{1}{\hat{C}_0}\left(\frac{V_0}{2Z_0}\int_{T+kT_0}^{t}e^{\mu(s-T-kT_0)}ds+\frac{e^{-\beta}J_0}{2Z_0}\int_{T+kT_0}^{t}e^{\mu(s-T-kT_0)}ds+\sum_{n=1}^{m}\left|g_n^{(0)}\right|\int_{T+kT_0}^{t}U_{G_0}^n e^{n\mu(s-T-kT_0)}\right)\le$$

$$\le \frac{V_0}{2Z_0\hat{C}_0}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}+\frac{e^{-\beta}J_0}{2Z_0\hat{C}_0}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}+\frac{1}{\hat{C}_0}\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n\frac{e^{\mu(t-T-kT_0)}-1}{n\mu}\le$$

$$\le \frac{V_0}{2Z_0\hat{C}_0}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}+\frac{e^{-\beta}J_0}{2Z_0\hat{C}_0}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}+\frac{1}{\hat{C}_0}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu T_0}\le$$

$$\le e^{\mu(t-T-kT_0)}\frac{1}{\mu\hat{C}_0}\left(\frac{V_0+e^{-\beta}J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right);$$

$$J_2 \le \frac{1}{\hat{C}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{|U(s)|}{2Z_0}+\frac{\left|\bar{I}_0(s)\right|}{2Z_0}+\left|G_0(u_{G_0C_0}(s))\right|\right)ds \le$$

$$\le \frac{1}{\hat{C}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{V_0 e^{\mu(s-T-kT_0)}}{2Z_0}+\frac{e^{-\beta}J_0 e^{\mu(s-T-kT_0)}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|\left|u_{G_0C_0}(s)\right|^n\right)ds \le$$

$$\le \frac{1}{\hat{C}_0}\left(\frac{V_0}{2Z_0}\int_{T+kT_0}^{T+(k+1)T_0}e^{\mu(s-T-kT_0)}ds+\frac{e^{-\beta}J_0}{2Z_0}\int_{T+kT_0}^{T+(k+1)T_0}e^{\mu(s-T-kT_0)}ds+\sum_{n=1}^{m}\left|g_n^{(0)}\right|\int_{T+kT_0}^{T+(k+1)T_0}U_{G_0}^n e^{n\mu(s-T-kT_0)}\right)\le$$

$$\le \frac{V_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{e^{-\beta}J_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{1}{\hat{C}_0}\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n\frac{e^{n\mu T_0}-1}{n\mu}\le$$

$$\le \frac{V_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{e^{-\mu T}J_0}{2Z_0\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}+\frac{1}{\hat{C}_0}\frac{e^{\mu T_0}-1}{\mu}\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu T_0}\le$$

$$\le e^{\mu(t-T-kT_0)}\frac{e^{\mu_0}-1}{\mu\hat{C}_0}\left(\frac{V_0+e^{-\beta}J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right).$$

Then

$$\left|B_0^{(k)}(u_{G_0C_0},U,I)(t)\right|\le J_1+J_2\le$$

$$\le e^{\mu(t-T-kT_0)}\frac{1}{\mu\hat{C}_0}\left(\frac{V_0+e^{-\beta}J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right)+e^{\mu(t-T-kT_0)}\frac{e^{\mu_0}-1}{\mu\hat{C}_0}\left(\frac{V_0+e^{-\beta}J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right)\le$$

$$\le e^{\mu(t-T-kT_0)}\frac{e^{\mu_0}}{\mu\hat{C}_0}\left(\frac{V_0+e^{-\beta}J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right)\le e^{\mu(t-T-kT_0)}U_{G_0}.$$

Further on, we have

$$\left|B_U^{(k)}(u_{G_0C_0},U,I)(t)\right| \le \left|\int_{T+kT_0}^{t} U(s)ds\right| + \left|\int_{T+kT_0}^{T+(k+1)T_0} U(s)ds\right| \le W_1 + W_2 .$$

Now

$$W_1 \le \left|\int_{T+kT_0}^{t} \frac{d\bar{I}_0(s)}{ds}ds\right| + \frac{2Z_0}{\hat{L}_0}\int_{T+kT_0}^{t}\left(\frac{|U(s)|}{2} + \frac{|\bar{I}_0(s)|}{2} + |u_{G_0C_0}(s)| + |E_0(s)|\right)ds \le$$

$$\le \left|\bar{I}_0(t)\right| + \frac{2Z_0}{\hat{L}_0}\int_{T+kT_0}^{t}\left(\frac{V_0}{2} + \frac{J_0e^{-\beta}}{2} + U_{G_0} + U_{E_0}\right)e^{\mu(s-T-kT_0)}ds \le$$

$$\le e^{\mu(t-T-kT_0)}J_0e^{-\beta} + \frac{2Z_0}{\hat{L}_0}\left(\frac{V_0}{2} + \frac{J_0e^{-\beta}}{2} + U_{G_0} + U_{E_0}\right)\frac{e^{\mu(t-T-kT_0)}-1}{\mu} \le$$

$$\le e^{\mu(t-T-kT_0)}\left[J_0e^{-\beta} + \frac{Z_0\left(V_0 + J_0e^{-\beta} + 2U_{G_0} + 2U_{E_0}\right)}{\mu\hat{L}_0}\right];$$

$$W_2 \le \left|\int_{T+kT_0}^{T+(k+1)T_0} \frac{d\bar{I}_0(s)}{ds}ds\right| + \frac{2Z_0}{\hat{L}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{|U(s)|}{2} + \frac{|\bar{I}_0(s)|}{2} + |u_{G_0C_0}(s)| + |E_0(s)|\right)ds \le$$

$$\le \frac{2Z_0}{\hat{L}_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(\frac{V_0}{2} + \frac{J_0e^{-\beta}}{2} + U_{G_0} + U_{E_0}\right)e^{\mu(s-T-kT_0)}ds \le e^{\mu(t-T-kT_0)}\frac{e^{\mu T_0}-1}{\mu}\frac{Z_0\left(V_0 + J_0e^{-\beta} + 2U_{G_0} + 2U_{E_0}\right)}{\hat{L}_0}.$$

Then

$$\left|B_U^{(k)}(u_{G_0C_0},U,I)(t)\right| \le W_1 + W_2 \le$$

$$\le e^{\mu(t-T-kT_0)}\left[J_0e^{-\beta} + \frac{1}{\mu}\frac{Z_0\left(V_0 + J_0e^{-\beta} + 2U_{G_0} + 2U_{E_0}\right)}{\hat{L}_0}\right] + e^{\mu(t-T-kT_0)}\frac{e^{\mu T_0}-1}{\mu}\frac{Z_0\left(V_0 + J_0e^{-\beta} + 2U_{G_0} + 2U_{E_0}\right)}{\hat{L}_0} \le$$

$$\le e^{\mu(t-T-kT_0)}\left[J_0e^{-\beta} + \frac{e^{\mu_0}}{\mu}\frac{Z_0\left(V_0 + J_0e^{-\beta} + 2U_{G_0} + 2U_{E_0}\right)}{\hat{L}_0}\right] \le V_0e^{\mu(t-T-kT_0)}.$$

Since

$$\left|B_1^{(k)}(U,I,u_{G_1C_1})(t)\right| \le \left|\int_{T+kT_0}^{t} U_{G_1}(s)ds\right| + \left|\int_{T+kT_0}^{T+(k+1)T_0} U_{G_1}(s)ds\right| \equiv P_1 + P_2$$

and

$$P_1 \le \frac{1}{\hat{C}_1}\int_{T+kT_0}^{t}\left(\frac{\bar{U}_0(s)}{2Z_0} + \frac{I(s)}{2Z_0} + G_1(u_{G_1C_1}(s))\right)ds \le e^{\mu(t-T-kT_0)}\frac{1}{\mu\hat{C}_1}\left(\frac{V_0e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m}\left|g_n^{(1)}\right|U_{G_1}^n e^{(n-1)\mu_0}\right);$$

$$P_2 \le \frac{1}{\hat{C}_1} \int_{T+kT_0}^{T+(k+1)T_0} \left(\frac{\vec{U}_0(s)}{2Z_0} + \frac{I(s)}{2Z_0} + G_1(u_{G_1C_1}(s)) \right) ds \le e^{\mu(t-T-kT_0)} \frac{e^{\mu_0}-1}{\mu \hat{C}_1} \left(\frac{V_0 e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(1)} \right| U_{G_1}^n e^{(n-1)\mu_0} \right)$$

we have

$$\left| B_1^{(k)}(U, I, u_{G_1C_1})(t) \right| \le P_1 + P_2 \le$$

$$\le e^{\mu(t-T-kT_0)} \frac{1}{\mu \hat{C}_1} \left(\frac{V_0 e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(1)} \right| U_{G_1}^n e^{(n-1)\mu_0} \right) + e^{\mu(t-T-kT_0)} \frac{e^{\mu_0}-1}{\mu \hat{C}_1} \left(\frac{V_0 e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(1)} \right| U_{G_1}^n e^{(n-1)\mu_0} \right) \le$$

$$\le e^{\mu(t-T-kT_0)} \frac{e^{\mu_0}}{\mu \hat{C}_1} \left(\frac{V_0 e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(1)} \right| U_{G_1}^n e^{(n-1)\mu_0} \right) \le e^{\mu(t-T-kT_0)} U_{G_1}.$$

On the other hand,

$$\left| B_I^{(k)}(U, I, u_{G_1C_1})(t) \right| \le \left| \int_{T+kT_0}^{t} I(s) ds \right| + \left| \int_{T+kT_0}^{T+(k+1)T_0} I(s) ds \right| \equiv I_1 + I_2$$

and

$$I_1 \le \left| \int_{T+kT_0}^{t} \frac{d\vec{U}_0(s)}{ds} ds \right| + \frac{2Z_0}{\hat{L}_1} \int_{T+kT_0}^{t} \left(\frac{\left| \vec{U}_0(s) \right|}{2} + \frac{\left| I(s) \right|}{2} + \left| u_{G_1C_1}(s) \right| + \left| E_1(s) \right| \right) ds \le$$

$$\le \left| \vec{U}_0(t) \right| + e^{\mu(t-T-kT_0)} \frac{Z_0 \left(V_0 e^{-\beta} + J_0 + 2U_{G_1} + 2U_{E_1} \right)}{\mu \hat{L}_1} \le$$

$$\le e^{\mu(t-T-kT_0)} \left[V_0 e^{-\beta} + \frac{1}{\mu} \frac{Z_0 \left(V_0 e^{-\beta} + J_0 + 2U_{G_1} + 2U_{E_1} \right)}{\hat{L}_1} \right];$$

$$I_2 \le \left| \int_{T+kT_0}^{T+(k+1)T_0} \frac{d\vec{U}_0(s)}{ds} ds \right| + \frac{2Z_0}{\hat{L}_1} \int_{T+kT_0}^{T+(k+1)T_0} \left(\frac{\left| \vec{U}_0(s) \right|}{2} + \frac{\left| I(s) \right|}{2} + \left| u_{G_1C_1}(s) \right| + \left| E_1(s) \right| \right) ds \le$$

$$\le e^{\mu(t-T-kT_0)} \frac{e^{\mu T_0}-1}{\mu} \frac{Z_0 \left(V_0 e^{-\beta} + J_0 + 2U_{G_1} + 2U_{E_1} \right)}{\hat{L}_1}.$$

Then

$$\left| B_I^{(k)}(U, I, u_{G_1C_1})(t) \right| \le I_1 + I_2 \le$$

$$\le e^{\mu(t-T-kT_0)} \left[V_0 e^{-\beta} + \frac{1}{\mu} \frac{Z_0 \left(V_0 e^{-\beta} + J_0 + 2U_{G_1} + 2U_{E_1} \right)}{\hat{L}_1} \right] + e^{\mu(t-T-kT_0)} \frac{e^{\mu_0}-1}{\mu} \frac{Z_0 \left(V_0 e^{-\beta} + J_0 + 2U_{G_1} + 2U_{E_1} \right)}{\hat{L}_1} \le$$

$$\le e^{\mu(t-T-kT_0)} \left[V_0 e^{-\beta} + \frac{e^{\mu_0}}{\mu} \frac{Z_0 \left(V_0 e^{-\beta} + J_0 + 2U_{G_1} + 2U_{E_1} \right)}{\hat{L}_1} \right] \le e^{\mu(t-T-kT_0)} J_0.$$

We use the results from Subsection 2.5. Then

$$\left|B_0^{(k)}(u_{G_0C_0},U,I)(t)-B_0^{(k)}(\bar{u}_{G_0C_0},\bar{U},\bar{I})(t)\right| \le \int_{T+kT_0}^{t}\left|U_{G_0}(u_{G_0C_0},U,I)(s)-U_{G_0}(\bar{u}_{G_0C_0},\bar{U},\bar{I})(s)\right|ds+$$

$$+\left|\int_{T+kT_0}^{T+(k+1)T_0}\left(U_{G_0}(u_{G_0C_0},U,I)(s)-U_{G_0}(\bar{u}_{G_0C_0},\bar{U},\bar{I})(s)\right)ds\right| \equiv K_1+K_2\,.$$

But

$$K_1 \le \int_{T+kT_0}^{t}\left(\left|\frac{\partial U_{G_0}(u_{G_0C_0},U,I)}{\partial u_{G_0C_0}}\right|\left|u_{G_0C_0}(s)-\bar{u}_{G_0C_0}(s)\right|+\left|\frac{\partial U_{G_0}(u_{G_0C_0},U,I)}{\partial U}\right|\left|U(s)-\bar{U}(s)\right|\right)ds \le$$

$$\le \int_{T+kT_o}^{t}\frac{1}{\left(d\tilde{C}_0(u_{G_0C_0})/du_{G_0C_0}\right)^2}\left|\frac{d^2\tilde{C}_0(u_{G_0C_0})}{du_{G_0C_0}^2}\right|\left(\frac{|U(s)|+|\bar{I}_0(s)|}{2Z_0}+\left|G_0(u_{G_0C_0}(s))\right|\right)\left|u_{G_0C_0}(s)-\bar{u}_{G_0C_0}(s)\right|ds+$$

$$+\int_{T+kT_o}^{t}\left|\frac{1}{d\tilde{C}_0(u_{G_0C_0})/du_{G_0C_0}}\right|\left|\frac{dG_0(u_{G_0C_0}(s))}{du_{G_0C_0})}\right|\left|u_{G_0C_0}(s)-\bar{u}_{G_0C_0}(s)\right|ds+$$

$$+\frac{1}{2Z_0}\int_{T+kT_0}^{t}\left|\frac{1}{d\tilde{C}_0(u_{G_0C_0}(s))/du_{G_0C_0}}\right|\left|U(s)-\bar{U}(s)\right|ds \le$$

$$\le \frac{1}{2}A_0^2B_0\int_{T+kT_o}^{t}\left(e^{\mu(s-T-kT_0)}\frac{V_0+J_0e^{-\beta}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^{n}e^{n\mu(s-T-kT_0)}\right)\left|u_{G_0C_0}(s)-\bar{u}_{G_0C_0}(s)\right|ds+$$

$$+A_0\int_{T+kT_o}^{t}\sum_{n=1}^{m-1}n\left|g_n^{(0)}\right|\left|u_{G_0C_0}(s)\right|^{n-1}\left|u_{G_0C_0}(s)-\bar{u}_{G_0C_0}(s)\right|ds+\frac{1}{2Z_0}A_0\int_{T+kT_0}^{t}\left|U(s)-\bar{U}(s)\right|ds \le$$

$$\le \frac{1}{2}A_0^2B_0\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^{n}e^{(n-1)\mu T_0}\right)\rho_\mu^{(k)}(u_{G_0C_0},\bar{u}_{G_0C_0})\int_{T+kT_o}^{t}e^{\mu(s-T-kT_0)}ds+$$

$$+A_0\sum_{n=1}^{m-1}n\left|g_n^{(0)}\right|\left|u_{G_0C_0}(s)\right|^{n-1}\rho_\mu^{(k)}(u_{G_0C_0},\bar{u}_{G_0C_0})\int_{T+kT_O}^{t}e^{\mu(s-T-kT_0)}ds+\frac{1}{2Z_0}A_0\rho_\mu^{(k)}(U,\bar{U})\int_{T+kT_O}^{t}e^{\mu(s-T-kT_0)}ds \le$$

$$\le \frac{e^{\mu(t-T-kT_0)}-1}{\mu}A_0\left[A_0B_0\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^{n}e^{(n-1)\mu T_0}\right)\frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0},\dot{\bar{u}}_{G_0C_0})}{\mu}+\right.$$

$$\left.+\sum_{n=1}^{m-1}n\left|g_n^{(0)}\right|\left|u_{G_0C_0}(s)\right|^{n-1}\frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0},\dot{\bar{u}}_{G_0C_0})}{\mu}+\frac{1}{2Z_0}\frac{\rho_\mu^{(k)}(\dot{U},\dot{\bar{U}})}{\mu}\right] \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\bar{U},\bar{u}_{G_1C_1},\bar{I}))\times$$

$$\times\frac{1}{\mu^2}A_0\left[\frac{1}{2}A_0B_0\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^{n}e^{(n-1)\mu T_0}\right)+\sum_{n=1}^{m-1}n\left|g_n^{(0)}\right|U_{G_0}{}^{n-1}+\frac{1}{2Z_0}\right];$$

$$K_2 \le \frac{1}{2} A_0^2 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) \rho_\mu^{(k)} (u_{G_0C_0}, \bar{u}_{G_0C_0}) \int_{T+kT_o}^{T+(k+1)T_o} e^{\mu(s-T-kT_0)} ds +$$

$$+ A_0 \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| \left| u_{G_0C_0}(s) \right|^{n-1} \rho_\mu^{(k)} (u_{G_0C_0}, \bar{u}_{G_0C_0}) \int_{T+kT_O}^{T+(k+1)T_O} e^{\mu(s-T-kT_0)} ds + \frac{1}{2Z_0} A_0 \rho_\mu^{(k)} (U, \overline{U}) \int_{T+kT_O}^{T+(k+1)T_O} e^{\mu(s-T-kT_0)} ds \le$$

$$\le \frac{e^{\mu T_0} - 1}{\mu} A_0 \left[\frac{1}{2} A_0 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) \frac{\rho_\mu^{(k)} (\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0})}{\mu} + \right.$$

$$\left. + \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| \left| u_{G_0C_0}(s) \right|^{n-1} \frac{\rho_\mu^{(k)} (\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0})}{\mu} + \frac{1}{2Z_0} \frac{\rho_\mu^{(k)} (\dot{U}, \dot{\overline{U}})}{\mu} \right] \le$$

$$\le e^{\mu(t-T-kT_0)} \rho_\mu ((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \overline{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{e^{\mu_0} - 1}{\mu^2} A_0 \left[\frac{1}{2} A_0 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) + \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| U_{G_0}{}^{n-1} + \frac{1}{2Z_0} \right].$$

Therefore,

$$\left| B_0^{(k)} (u_{G_0C_0}, U, I)(t) - B_0^{(k)} (\bar{u}_{G_0C_0}, \overline{U}, \bar{I})(t) \right| \le$$

$$\le e^{\mu T_0} \rho_\mu ((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \overline{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{e^{\mu_0}}{\mu^2} A_0 \left[\frac{1}{2} A_0 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) + \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| U_{G_0}{}^{n-1} + \frac{1}{2Z_0} \right]$$

implies that

$$\hat{\rho}(B_0(u_{G_0C_0}, U, I), B_0(\bar{u}_{G_0C_0}, \overline{U}, \bar{I})) \le$$

$$\le e^{\mu T_0} \rho_\mu ((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \overline{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{e^{\mu_0}}{\mu^2} A_0 \left[\frac{1}{2} A_0 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) + \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| U_{G_0}{}^{n-1} + \frac{1}{2Z_0} \right] \equiv$$

$$\equiv e^{\mu T_0} K_0 \rho_\mu ((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \overline{U}, \bar{u}_{G_1C_1}, \bar{I})) .$$

For the second component, we have

$$\left| B_U^{(k)} (u_{G_0C_0}, U, I)(t) - B_U^{(k)} (\bar{u}_{G_0C_0}, \overline{U}, \bar{I})(t) \right| \le \left| \int_{T+kT_0}^{t} \left(U(u_{G_0C_0}, U, I)(s) - U(\bar{u}_{G_0C_0}, \overline{U}, \bar{I})(s) \right) ds \right| +$$

$$+ \left| \int_{T+kT_0}^{T+(k+1)T_0} \left(U(u_{G_0C_0}, U, I)(s) - U(\bar{u}_{G_0C_0}, \overline{U}, \bar{I})(s) \right) ds \right| \le W_1 + W_2 .$$

Now

$$W_1 \le \int_{T+kT_0}^{t} \left| \frac{\partial U(u_{G_0C_0}, U, I)}{\partial u_{G_0C_0}} \right| \left| u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s) \right| ds + \int_{T+kT_0}^{t} \left| \frac{\partial U(u_{G_0C_0}, U, I)}{\partial U} \right| \left| U(s) - \bar{U}(s) \right| ds \le$$

$$\le \frac{2Z_0}{\hat{L}_0} \int_{T+kT_0}^{t} \left| u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s) \right| ds +$$

$$+ \left[\frac{1}{\hat{L}_0^2} \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) + \frac{Z_0}{\hat{L}_0} \right] \int_{T+kT_0}^{t} \left| U(s) - \bar{U}(s) \right| ds \le$$

$$\le \frac{2Z_0}{\hat{L}_0} \rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) \int_{T+kT_0}^{t} e^{\mu(s-T-kT_0)} ds +$$

$$+ \left[\frac{1}{\hat{L}_0^2} \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) + \frac{Z_0}{\hat{L}_0} \right] \rho_\mu^{(k)} U, \bar{U}) \int_{T+kT_0}^{t} e^{\mu(s-T-kT_0)} ds \le$$

$$\le \frac{2Z_0}{\hat{L}_0} \frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0})}{\mu} \frac{e^{\mu(t-T-kT_0)} - 1}{\mu} +$$

$$+ \left[\frac{1}{\hat{L}_0^2} \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) + \frac{Z_0}{\hat{L}_0} \right] \frac{\rho_\mu^{(k)} \dot{U}, \dot{\bar{U}})}{\mu} \frac{e^{\mu(t-T-kT_0)} - 1}{\mu} \le$$

$$\le e^{\mu(t-T-kT_0)} \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{1}{\mu^2} \left[\frac{2Z_0}{\hat{L}_0} + \frac{1}{\hat{L}_0^2} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} + \frac{Z_0}{\hat{L}_0} \right];$$

$$W_2 \le \left| \int_{T+kT_0}^{T+(k+1)T_0} \left(U(u_{G_0C_0}, U, I)(s) - U(\bar{u}_{G_0C_0}, \bar{U}, \bar{I})(s) \right) ds \right| \le \frac{2Z_0}{\hat{L}_0} \int_{T+kT_0}^{T+(k+1)T_0} \left| u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s) \right| ds +$$

$$+ \left[\frac{1}{\hat{L}_0^2} \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) + \frac{Z_0}{\hat{L}_0} \right] \int_{T+kT_0}^{T+(k+1)T_0} \left| U(s) - \bar{U}(s) \right| ds \le$$

$$\le e^{\mu(t-T-kT_0)} \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{e^{\mu_0} - 1}{\mu^2} \left[\frac{2Z_0}{\hat{L}_0} + \frac{1}{\hat{L}_0^2} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} + \frac{Z_0}{\hat{L}_0} \right].$$

Then

$$\left| B_U^{(k)}(u_{G_0C_0}, U, I)(t) - B_U^{(k)}(\bar{u}_{G_0C_0}, \bar{U}, \bar{I})(t) \right| \le$$

$$\leq e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\bar{U},\bar{u}_{G_1C_1},\bar{I}))\times$$
$$\times\frac{e^{\mu_0}}{\mu^2}\left[\frac{3Z_0}{\hat{L}_0}+\frac{1}{\hat{L}_0^2}\left(\frac{V_0+J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)\sum_{n=1}^{m}(n+1)nl_n^{(0)}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}\right)^{n-1}\right]$$

and

$$\hat{\rho}(B_U(u_{G_0C_0},U,I),B_0(\bar{u}_{G_0C_0},\bar{U},\bar{I}))\leq$$
$$\leq e^{\mu T_0}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\bar{U},\bar{u}_{G_1C_1},\bar{I}))\times$$
$$\times\frac{e^{\mu_0}}{\mu^2}\left[\frac{3Z_0}{\hat{L}_0}+\frac{1}{\hat{L}_0^2}\left(\frac{V_0+J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)\sum_{n=1}^{m}(n+1)nl_n^{(0)}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}\right)^{n-1}\right]\equiv$$
$$\equiv e^{\mu_0}K_U\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\bar{U},\bar{u}_{G_1C_1},\bar{I})).$$

For the third component, we have

$$\left|B_1^{(k)}(U,I,u_{G_1C_1})(t)-B_1^{(k)}(\bar{U},\bar{I},\bar{u}_{G_1C_1})(t)\right|\leq\left|\int_{T+kT_0}^{t}\left(U_{G_1}(U,I,u_{G_1C_1})(s)-U_{G_1}(\bar{U},\bar{I},\bar{u}_{G_1C_1})(s)\right)ds\right|+$$
$$+\left|\int_{T+kT_0}^{T+(k+1)T_0}\left(U_{G_1}(U,I,u_{G_1C_1})(s)-U_{G_1}(\bar{U},\bar{I},\bar{u}_{G_1C_1})(s)\right)ds\right|\equiv S_1+S_2.$$

But

$$S_1\leq\int_{T+kT_0}^{t}\left|\frac{\partial U_{G_1}(U,u_{G_1C_1},I)(s)}{\partial u_{G_1C_1}}\right|\left|u_{G_1C_1}(s)-\bar{u}_{G_1C_1}(s)\right|ds+\int_{T+kT_0}^{t}\left|\frac{\partial U_{G_1}(U,u_{G_1C_1},I)(s)}{\partial I}\right|\left|I(s)-\bar{I}(s)\right|ds\leq$$
$$\leq\left[A_1^2B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}\right]\int_{T+kT_0}^{t}\left|u_{G_1C_1}(s)-\bar{u}_{G_1C_1}(s)\right|ds+$$
$$+\frac{1}{2Z_0}A_1\int_{T+kT_0}^{t}\left|I(s)-\bar{I}(s)\right|ds\leq$$
$$\leq\left[A_1^2B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}\right]\rho_\mu^{(k)}(u_{G_1C_1},\bar{u}_{G_1C_1})\times$$
$$\times\int_{T+kT_0}^{t}e^{\mu(s-T-kT_0)}ds+\frac{1}{2Z_0}A_1\rho_\mu^{(k)}(I,\bar{I})\int_{T+kT_0}^{t}e^{\mu(s-T-kT_0)}ds\leq$$

$$\leq\left[A_1^{\,2}B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}\right]\frac{\rho_\mu^{(k)}(\dot{u}_{G_1C_1},\dot{\bar{u}}_{G_1C_1})}{\mu}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}+$$

$$+\frac{1}{2Z_0}A_1\frac{\rho_\mu^{(k)}(\dot{I},\dot{\bar{I}})}{\mu}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}\leq$$

$$\leq e^{\mu(t-T-kT_0)}\frac{1}{\mu^2}\left[\left(\frac{\sqrt[h]{\Phi_1+\phi_0}}{c_1\sqrt[h]{\Phi_1}\,\phi_0}\right)^2\frac{2c_1\sqrt[h]{\Phi_1}\left(h\Phi_1+\phi_0\right)}{h^2\ \sqrt[h]{\left(\Phi_1-\phi_0\right)^{1+2h}}}\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+\right.$$

$$\left.\frac{\sqrt[h]{\Phi_1+\phi_0}}{c_1\sqrt[h]{\Phi_1}\,\phi_0}\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}+\frac{1}{2Z_0}\frac{\sqrt[h]{\Phi_1+\phi_0}}{c_1\sqrt[h]{\Phi_1}\,\phi_0}\right]\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}));$$

$$S_2\leq\int_{T+kT_0}^{T+(k+1)T_0}\left|\frac{\partial U_{G_1}(U,u_{G_1C_1},I)(s)}{\partial u_{G_1C_1}}\right|\left|u_{G_1C_1}(s)-\bar{u}_{G_1C_1}(s)\right|ds+\int_{T+kT_0}^{T+(k+1)T_0}\left|\frac{\partial U_{G_1}(U,u_{G_1C_1},I)(s)}{\partial I}\right|\left|I(s)-\bar{I}(s)\right|ds\leq$$

$$\leq e^{\mu(t-T-kT_0)}\frac{e^{\mu_0}-1}{\mu^2}\left[A_1^{\,2}B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+\right.$$

$$\left.+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}+\frac{1}{2Z_0}A_1\right]\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I})).$$

Then

$$\left|B_1^{(k)}(U,I,u_{G_1C_1})(t)-B_1^{(k)}(\overline{U},\bar{I},\bar{u}_{G_1C_1})(t)\right|\leq$$

$$\leq e^{\mu(t-T-kT_0)}\frac{e^{\mu_0}A_1}{\mu^2}\left[A_1B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+\right.$$

$$\left.+\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}+\frac{1}{2Z_0}\right]\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))\equiv$$

$$\equiv e^{\mu(t-T-kT_0)}K_1\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))\leq$$

$$\leq e^{\mu_0}K_1\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))\,.$$

It follows that

$$\hat{\rho}(B_1(U,u_{G_1C_1},I),B_1(\overline{U},\bar{u}_{G_1C_1},\bar{I}))\leq e^{\mu_0}K_1\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))\,.$$

For the fourth component, we have

$$\left|B_I^{(k)}(U,u_{G_1C_1},I)(t)-B_I^{(k)}(\overline{U},\bar{u}_{G_1C_1},\bar{I})(t)\right|\leq\left|\int_{T+kT_0}^{t}\left(I(U,u_{G_1C_1},I)-I(\overline{U},\bar{u}_{G_1C_1},\bar{I})(s)\right)ds\right|+$$

$$+\left|\int_{T+kT_0}^{T+(k+1)T_0}\left(I(U,u_{G_1C_1},I)-I(\overline{U},\overline{u}_{G_1C_1},\overline{I})(s)\right)ds\right| \equiv Q_1+Q_2 .$$

But

$$Q_1 \le \frac{2Z_0}{\hat{L}_1}\int_{T+kT_0}^{t}\left|u_{G_1C_1}(s)-\overline{u}_{G_1C_1}(s)\right|ds+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\int_{T+kT_0}^{t}\left|I(s)-\overline{I}(s)\right|ds \le$$

$$\le \frac{2Z_0}{\hat{L}_1}\rho_\mu^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1})\int_{T+kT_0}^{t}e^{\mu(s-T-kT_0)}ds+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\rho_\mu^{(k)}(I,\overline{I})\int_{T+kT_0}^{t}e^{\mu(s-T-kT_0)}ds \le$$

$$\le \frac{2Z_0}{\hat{L}_1}\frac{\rho_\mu^{(k)}(\dot{u}_{G_1C_1},\dot{\overline{u}}_{G_1C_1})}{\mu}\frac{e^{\mu(t-T-kT_0)}-1}{\mu}+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\frac{\rho_\mu^{(k)}(\dot{I},\dot{\overline{I}})}{\mu}\frac{e^{\mu(t-T-kT_0)}-1}{\mu} \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))\times$$

$$\times\frac{1}{\mu^2}\left[\frac{3Z_0}{\hat{L}_1}+\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\mu T}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\mu T}+J_0}{2}+U_{G_1}+U_{E_1}\right)\right];$$

$$Q_2 \le \frac{2Z_0}{\hat{L}_1}\rho_\mu^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1})\int_{T+kT_0}^{T+(k+1)T_0}e^{\mu(s-T-kT_0)}ds+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\rho_\mu^{(k)}(I,\overline{I})\int_{T+kT_0}^{T+(k+1)T_0}e^{\mu(s-T-kT_0)}ds \le$$

$$\le \frac{2Z_0}{\hat{L}_1}\frac{\rho_\mu^{(k)}(\dot{u}_{G_1C_1},\dot{\overline{u}}_{G_1C_1})}{\mu}\frac{e^{\mu T_0}-1}{\mu}+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\frac{\rho_\mu^{(k)}(\dot{I},\dot{\overline{I}})}{\mu}\frac{e^{\mu T_0}-1}{\mu} \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))\times$$

$$\times\frac{e^{\mu_0}-1}{\mu^2}\left[\frac{3Z_0}{\hat{L}_1}+\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)\right].$$

It follows that

$$\left|B_I^{(k)}(U,I,u_{G_1C_1})(t)-B_I^{(k)}(\overline{U},\bar{I},\bar{u}_{G_1C_1})(t)\right|\le$$
$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))\times$$
$$\times\frac{e^{\mu_0}}{\mu^2}\left[\frac{3Z_0}{\hat{L}_1}+\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)\right]\equiv$$
$$\equiv e^{\mu(t-T-kT_0)}K_I\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))\le$$
$$\le e^{\mu_0}K_I\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))$$

or

$$\hat{\rho}(B_I(U,I,u_{G_1C_1}),B_I(\overline{U},\bar{I},\bar{u}_{G_1C_1}))\le e^{\mu_0}K_I\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}).$$

For the derivatives, we obtain

$$\left|\dot{B}_0^{(k)}(u_{G_0C_0},U,I)(t)-\dot{B}_0^{(k)}(\bar{u}_{G_0C_0},\overline{U},\bar{I})(t)\right|\le\left|U_{G_0}(u_{G_0C_0},U,I)(t)-U_{G_0}(\bar{u}_{G_0C_0},\overline{U},\bar{I})(t)\right|+$$
$$+\frac{1}{T_0}\left|\int_{T+kT_0}^{T+(k+1)T_0}\left(U_{G_0}(u_{G_0C_0},U,I)(s)-U_{G_0}(\bar{u}_{G_0C_0},\overline{U},\bar{I})(s)\right)ds\right|\equiv\dot{I}_1+\dot{I}_2.$$

Since

$$\dot{I}_1\le\left|\frac{\partial U_{G_0}(u_{G_0C_0},U,I)}{\partial u_{G_0C_0}}\right|\left|u_{G_0C_0}(t)-\bar{u}_{G_0C_0}(t)\right|+\left|\frac{\partial U_{G_0}(u_{G_0C_0},U,I)}{\partial U}\right|\left|U(t)-\overline{U}(t)\right|\le$$
$$\le\frac{1}{\left(d\widetilde{C}_0(u_{G_0C_0})/du_{G_0C_0}\right)^2}\left|\frac{d^2\widetilde{C}_0(u_{G_0C_0})}{du_{G_0C_0}^2}\right|\left(\frac{|U(t)|+|\bar{I}_0(t)|}{2Z_0}+\left|G_0(u_{G_0C_0}(t))\right|\right)\left|u_{G_0C_0}(t)-\bar{u}_{G_0C_0}(t)\right|+$$
$$+\left|\frac{1}{d\widetilde{C}_0(u_{G_0C_0})/du_{G_0C_0}}\right|\left|\frac{dG_0(u_{G_0C_0}(t))}{du_{G_0C_0})}\right|\left|u_{G_0C_0}(t)-\bar{u}_{G_0C_0}(t)\right|+$$
$$+\frac{1}{2Z_0}\left|\frac{1}{d\widetilde{C}_0(u_{G_0C_0}(s))/du_{G_0C_0}}\right|\left|U(t)-\overline{U}(t)\right|\le$$

$$\leq \frac{1}{2} A_0^2 B_0 \left(e^{\mu(t-T-kT_0)} \frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{n\mu T_0} \right) \left| u_{G_0C_0}(t) - \bar{u}_{G_0C_0}(t) \right| +$$

$$+ A_0 \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| \left| u_{G_0C_0}(t) \right|^{n-1} \left| u_{G_0C_0}(t) - \bar{u}_{G_0C_0}(t) \right| + \frac{1}{2Z_0} A_0 \left| U(t) - \bar{U}(t) \right| \leq$$

$$\leq \frac{1}{2} A_0^2 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) \rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) +$$

$$+ A_0 \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| \left| u_{G_0C_0}(s) \right|^{n-1} \rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) + \frac{1}{2Z_0} A_0 \rho_\mu^{(k)}(U, \bar{U}) \leq$$

$$\leq \frac{1}{2} A_0^2 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) \frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0})}{\mu} +$$

$$+ A_0 \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| \left| u_{G_0C_0}(s) \right|^{n-1} \frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0})}{\mu} + \frac{1}{2Z_0} A_0 \frac{\rho_\mu^{(k)}(\dot{U}, \dot{\bar{U}})}{\mu} \leq$$

$$\leq e^{\mu(t-T-kT_0)} \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{A_0}{\mu} \left[\frac{1}{2} A_0 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) + \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| U_{G_0}{}^{n-1} + \frac{1}{2Z_0} \right];$$

$$\dot{I}_2 \leq \frac{1}{2} A_0^2 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) \rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) \frac{1}{T_0} \int_{T+kT_o}^{T+(k+1)T_o} e^{\mu(s-T-kT_0)} ds +$$

$$+ A_0 \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| U_{G_0C_0}{}^{n-1} \rho_\mu^{(k)}(u_{G_0C_0}, \bar{u}_{G_0C_0}) \frac{1}{T_0} \int_{T+kT_O}^{T+(k+1)T_O} e^{\mu(s-T-kT_0)} ds +$$

$$+ \frac{1}{2Z_0} A_0 \rho_\mu^{(k)}(U, \bar{U}) \frac{1}{T_0} \int_{T+kT_O}^{T+(k+1)T_O} e^{\mu(s-T-kT_0)} ds \leq$$

$$\leq \frac{e^{\mu T_0} - 1}{\mu T_0} A_0 \left[\frac{1}{2} A_0 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) \frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0})}{\mu} + \right.$$

$$\left. + \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| \left| u_{G_0C_0}(s) \right|^{n-1} \frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0}, \dot{\bar{u}}_{G_0C_0})}{\mu} + \frac{1}{2Z_0} \frac{\rho_\mu^{(k)}(\dot{U}, \dot{\bar{U}})}{\mu} \right] \leq$$

$$\leq e^{\mu(t-T-kT_0)} \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{e^{\mu T_0} - 1}{\mu T_0} \frac{A_0}{\mu} \left[\frac{1}{2} A_0 B_0 \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(0)} \right| U_{G_0}^n e^{(n-1)\mu T_0} \right) + \sum_{n=1}^{m-1} n \left| g_n^{(0)} \right| U_{G_0}{}^{n-1} + \frac{1}{2Z_0} \right].$$

Then

$$\left| \dot{B}_0^{(k)}(u_{G_0C_0}, U, I)(t) - \dot{B}_0^{(k)}(\bar{u}_{G_0C_0}, \bar{U}, \bar{I})(t) \right| \leq$$

$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))\times$$
$$\times\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{A_0}{\mu}\left[\frac{1}{2}A_0B_0\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^{\,n}e^{(n-1)\mu T_0}\right)+\sum_{n=1}^{m-1}n\left|g_n^{(0)}\right|U_{G_0}{}^{n-1}+\frac{1}{2Z_0}\right]\equiv$$
$$\equiv e^{\mu(t-T-kT_0)}\dot{K}_0\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I})).$$

Therefore,

$$\rho_\mu^{(k)}(\dot{B}_0^{(k)}(u_{G_0C_0},U,I),\dot{B}_0^{(k)}(\overline{u}_{G_0C_0},\overline{U},\overline{I}))\le\dot{K}_0\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}).$$

For the second component, we have

$$\left|\dot{B}_U^{(k)}(u_{G_0C_0},U,I)(t)-\dot{B}_U^{(k)}(\overline{u}_{G_0C_0},\overline{U},\overline{I})(t)\right|\le\left|U(u_{G_0C_0},U,I)(t)-U(\overline{u}_{G_0C_0},\overline{U},\overline{I})(t)\right|+$$
$$+\left|\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(U(u_{G_0C_0},U,I)(s)-U(\overline{u}_{G_0C_0},\overline{U},\overline{I})(s)\right)ds\right|\le\dot{W}_1+\dot{W}_2.$$

Now

$$\dot{W}_1\le\left|\frac{\partial U(u_{G_0C_0},U,I)}{\partial u_{G_0C_0}}\right|\left|u_{G_0C_0}(t)-\overline{u}_{G_0C_0}(t)\right|+\left|\frac{\partial U(u_{G_0C_0},U,I)}{\partial U}\right|\left|U(s)-\overline{U}(s)\right|\le$$
$$\le\frac{2Z_0}{\hat{L}_0}\left|u_{G_0C_0}(t)-\overline{u}_{G_0C_0}(t)\right|+$$
$$+\left[\frac{1}{\hat{L}_0^2}\sum_{n=1}^{m}(n+1)nl_n^{(0)}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}\right)^{n-1}e^{\mu T_0}\left(\frac{V_0+J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)+\frac{Z_0}{\hat{L}_0}\right]\left|U(t)-\overline{U}(t)\right|\le$$
$$\le e^{\mu(t-T-kT_0)}\frac{2Z_0}{\hat{L}_0}\rho_\mu^{(k)}(u_{G_0C_0},\overline{u}_{G_0C_0})+$$
$$+e^{\mu(t-T-kT_0)}\left[\frac{1}{\hat{L}_0^2}\sum_{n=1}^{m}(n+1)nl_n^{(0)}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}\right)^{n-1}e^{\mu T_0}\left(\frac{V_0+J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)+\frac{Z_0}{\hat{L}_0}\right]\rho_\mu^{(k)}U,\overline{U})\le$$
$$\le e^{\mu(t-T-kT_0)}\frac{2Z_0}{\hat{L}_0}\frac{\rho_\mu^{(k)}(\dot{u}_{G_0C_0},\dot{\overline{u}}_{G_0C_0})}{\mu}+$$
$$+e^{\mu(t-T-kT_0)}\left[\frac{1}{\hat{L}_0^2}\sum_{n=1}^{m}(n+1)nl_n^{(0)}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}\right)^{n-1}e^{\mu T_0}\left(\frac{V_0+J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)+\frac{Z_0}{\hat{L}_0}\right]\frac{\rho_\mu^{(k)}\dot{U},\dot{\overline{U}})}{\mu}\le$$
$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))\times$$
$$\times\frac{1}{\mu}\left[\frac{3Z_0}{\hat{L}_0}+\frac{e^{\mu T_0}}{\hat{L}_0^2}\left(\frac{V_0+J_0e^{-\beta}}{2}+U_{G_0}+U_{E_0}\right)\sum_{n=1}^{m}(n+1)nl_n^{(0)}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}\right)^{n-1}\right];$$

$$W_2 \le \left| \frac{1}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} \left(U(u_{G_0C_0}, U, I)(s) - U(\bar{u}_{G_0C_0}, \bar{U}, \bar{I})(s) \right) ds \right| \le \frac{2Z_0}{\hat{L}_0} \frac{1}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} \left| u_{G_0C_0}(s) - \bar{u}_{G_0C_0}(s) \right| ds +$$

$$+ \left[\frac{1}{\hat{L}_0^2} \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} e^{\mu T_0} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) + \frac{Z_0}{\hat{L}_0} \right] \int_{T+kT_0}^{T+(k+1)T_0} \left| U(s) - \bar{U}(s) \right| ds \le$$

$$\le e^{\mu(t-T-kT_0)} \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \frac{e^{\mu_0} - 1}{\mu_0} \frac{1}{\mu} \left[\frac{3Z_0}{\hat{L}_0} + \frac{e^{\mu T_0}}{\hat{L}_0^2} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} \right].$$

Then

$$\left| \dot{B}_U^{(k)}(u_{G_0C_0}, U, I)(t) - \dot{B}_U^{(k)}(\bar{u}_{G_0C_0}, \bar{U}, \bar{I})(t) \right| \le$$

$$\le e^{\mu(t-T-kT_0)} \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})) \times$$

$$\times \left(1 + \frac{e^{\mu_0} - 1}{\mu_0} \right) \frac{1}{\mu} \left[\frac{3Z_0}{\hat{L}_0} + \frac{e^{\mu_0}}{\hat{L}_0^2} \left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0} \right) \sum_{n=1}^{m} (n+1) n l_n^{(0)} \left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} \right)^{n-1} \right] \equiv$$

$$\equiv e^{\mu(t-T-kT_0)} \dot{K}_U \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})).$$

Therefore,

$$\rho_\mu^{(k)}(\dot{B}_U^{(k)}(u_{G_0C_0}, U, I), \dot{B}_U^{(k)}(\bar{u}_{G_0C_0}, \bar{U}, \bar{I})) \le \dot{K}_U \rho_\mu((u_{G_0C_0}, U, u_{G_1C_1}, I), (\bar{u}_{G_0C_0}, \bar{U}, \bar{u}_{G_1C_1}, \bar{I})).$$

For the third component

$$\left| \dot{B}_1^{(k)}(U, I, u_{G_1C_1})(t) - \dot{B}_1^{(k)}(\bar{U}, \bar{I}, \bar{u}_{G_1C_1})(t) \right| \le \left| U_{G_1}(U, I, u_{G_1C_1})(t) - U_{G_1}(\bar{U}, \bar{I}, \bar{u}_{G_1C_1})(t) \right| +$$

$$+ \left| \frac{1}{T_0} \int_{T+kT_0}^{T+(k+1)T_0} \left(U_{G_1}(U, I, u_{G_1C_1})(s) - U_{G_1}(\bar{U}, \bar{I}, \bar{u}_{G_1C_1})(s) \right) ds \right| \equiv \dot{D}_1 + \dot{D}_2.$$

But

$$\dot{D}_1 \le \left| \frac{\partial U_{G_1}(U, u_{G_1C_1}, I)(s)}{\partial u_{G_1C_1}} \right| \left| u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t) \right| + \left| \frac{\partial U_{G_1}(U, u_{G_1C_1}, I)(s)}{\partial I} \right| \left| I(t) - \bar{I}(t) \right| \le$$

$$\le \left[A_1^2 B_1 \left(e^{\mu_0} \frac{V_0 e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m} \left| g_n^{(1)} \right| \left(U_{G_1} \right)^n e^{n\mu_0} \right) + \right.$$

$$\left. + A_1 \sum_{n=1}^{m} n \left| g_n^{(1)} \right| \left(U_{G_1} \right)^{n-1} e^{(n-1)\mu_0} \right] \left| u_{G_1C_1}(t) - \bar{u}_{G_1C_1}(t) \right| + \frac{1}{2Z_0} A_1 \left| I(t) - \bar{I}(t) \right| \le$$

$$\leq\left[A_1^2B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}\right]\rho_\mu^{(k)}(u_{G_1C_1},\bar{u}_{G_1C_1})e^{\mu(t-T-kT_0)}+$$

$$+\frac{1}{2Z_0}A_1\rho_\mu^{(k)}(I,\bar{I})e^{\mu(t-T-kT_0)}\leq$$

$$\leq\left[A_1^2B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}\right]\frac{\rho_\mu^{(k)}(\dot{u}_{G_1C_1},\dot{\bar{u}}_{G_1C_1})}{\mu}e^{\mu(t-T-kT_0)}+$$

$$+\frac{1}{2Z_0}A_1\frac{\rho_\mu^{(k)}(\dot{I},\dot{\bar{I}})}{\mu}e^{\mu(t-T-kT_0)}\leq$$

$$\leq e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}))\times$$

$$\times\frac{A_1}{\mu}\left[A_1B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}+\frac{1}{2Z_0}\right];$$

$$\dot{D}_2\leq\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\left|\frac{\partial U_{G_1}(U,u_{G_1C_1},I)(s)}{\partial u_{G_1C_1}}\right|\left|u_{G_1C_1}(s)-\bar{u}_{G_1C_1}(s)\right|ds+\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\left|\frac{\partial U_{G_1}(U,u_{G_1C_1},I)(s)}{\partial I}\right|\left|I(s)-\bar{I}(s)\right|ds\leq$$

$$\leq e^{\mu(t-T-kT_0)}\frac{e^{\mu_0}-1}{\mu_0}\frac{1}{\mu}\left[\left(A_1\right)^2B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+\right.$$

$$\left.+A_1\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}+\frac{1}{2Z_0}A_1\right]\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I})).$$

Then

$$\left|\dot{B}_1^{(k)}(U,u_{G_1C_1},I)(t)-\dot{B}_1^{(k)}(\overline{U},\bar{u}_{G_1C_1},\bar{I})(t)\right|\leq$$

$$\leq e^{\mu(t-T-kT_0)}\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{A_1}{\mu}\left[A_1B_1\left(e^{\mu_0}\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right)+\right.$$

$$\left.+\sum_{n=1}^{m}n\left|g_n^{(1)}\right|\left(U_{G_1}\right)^{n-1}e^{(n-1)\mu_0}+\frac{1}{2Z_0}\right]\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I})).$$

Therefore,

$$\rho_\mu^{(k)}(\dot{B}_1^{(k)}(U,u_{G_1C_1},I),\dot{B}_1^{(k)}(\overline{U},\bar{u}_{G_1C_1},\bar{I}))\leq\dot{K}_1\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\bar{u}_{G_0C_0},\overline{U},\bar{u}_{G_1C_1},\bar{I}).$$

For the fourth component, we obtain

$$\left|\dot{B}_I^{(k)}(U,u_{G_1C_1},I)(t)-\dot{B}_I^{(k)}(\overline{U},\bar{u}_{G_1C_1},\bar{I})(t)\right|\leq\left|I(U,u_{G_1C_1},I)(t)-I(\overline{U},\bar{u}_{G_1C_1},\bar{I})(t)\right|+$$

$$+\left|\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}\left(I(U,u_{G_1C_1},I)-I(\overline{U},\overline{u}_{G_1C_1},\overline{I})(s)\right)ds\right| \equiv \dot{M}_1+\dot{M}_2\,.$$

But

$$\dot{M}_1 \le \frac{2Z_0}{\hat{L}_1}\left|u_{G_1C_1}(t)-\overline{u}_{G_1C_1}(t)\right|+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\left|I(t)-\overline{I}(t)\right| \le$$

$$\le \frac{2Z_0}{\hat{L}_1}\rho_\mu^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1})e^{\mu(t-T-kT_0)}+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\rho_\mu^{(k)}(I,\overline{I})e^{\mu(t-T-kT_0)} \le$$

$$\le e^{\mu(t-T-kT_0)}\,\frac{2Z_0}{\hat{L}_1}\frac{\rho_\mu^{(k)}(\dot{u}_{G_1C_1},\dot{\overline{u}}_{G_1C_1})}{\mu}+$$

$$+e^{\mu(t-T-kT_0)}\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\frac{\rho_\mu^{(k)}(\dot{I},\dot{\overline{I}})}{\mu} \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))\times$$

$$\times\frac{1}{\mu^2}\left[\frac{3Z_0}{\hat{L}_1}+\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)\right];$$

$$\dot{M}_2 \le \frac{2Z_0}{\hat{L}_1}\rho_\mu^{(k)}(u_{G_1C_1},\overline{u}_{G_1C_1})\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}e^{\mu(s-T-kT_0)}ds+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\rho_\mu^{(k)}(I,\overline{I})\frac{1}{T_0}\int_{T+kT_0}^{T+(k+1)T_0}e^{\mu(s-T-kT_0)}ds \le$$

$$\le \frac{2Z_0}{\hat{L}_1}\frac{\rho_\mu^{(k)}(\dot{u}_{G_1C_1},\dot{\overline{u}}_{G_1C_1})}{\mu}\frac{e^{\mu T_0}-1}{\mu T_0}+$$

$$+\left[\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)+\frac{Z_0}{\hat{L}_1}\right]\frac{\rho_\mu^{(k)}(\dot{I},\dot{\overline{I}})}{\mu}\frac{e^{\mu T_0}-1}{\mu T_0} \le$$

$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))\times$$

$$\times\frac{e^{\mu_0}-1}{\mu_0}\frac{1}{\mu}\left[\frac{3Z_0}{\hat{L}_1}+\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)\right].$$

It follows that

$$\left|\dot{B}_I^{(k)}(U,u_{G_1C_1},I)(t)-\dot{B}_I^{(k)}(\overline{U},\overline{u}_{G_1C_1},\overline{I})(t)\right|\le$$
$$\le e^{\mu(t-T-kT_0)}\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))\times$$
$$\times\left(1+\frac{e^{\mu_0}-1}{\mu_0}\right)\frac{1}{\mu}\left[\frac{3Z_0}{\hat{L}_1}+\frac{1}{\hat{L}_1^2}\sum_{n=1}^{m}(n+1)nl_n^{(1)}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}\right)^{n-1}e^{\mu_0}\left(\frac{V_0e^{-\beta}+J_0}{2}+U_{G_1}+U_{E_1}\right)\right]\equiv$$
$$\equiv e^{\mu(t-T-kT_0)}\dot{K}_I\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}))$$

or

$$\hat{\rho}(B_I(U,u_{G_1C_1},I),B_I(\overline{U},\overline{u}_{G_1C_1},\overline{I}))\le\dot{K}_I\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I}).$$

Introducing a denotation

$B_0(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I})\equiv\overline{B}_0$, $B_U(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I})\equiv\overline{B}_U$, $B_1(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I})\equiv\overline{B}_1$, $B_I(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I})\equiv\overline{B}_I$.

On the base of all of above inequalities, we obtain

$$\rho_\mu((B_0,B_U,B,B_I),(\overline{B}_0,\overline{B}_U,\overline{B},\overline{B}_I))\le K\rho_\mu((u_{G_0C_0},U,u_{G_1C_1},I),(\overline{u}_{G_0C_0},\overline{U},\overline{u}_{G_1C_1},\overline{I})$$

where $K=\max\left\{e^{\mu_0}K_0,e^{\mu_0}K_U,e^{\mu_0}K_1,e^{\mu_0}K_I,\dot{K}_0,\dot{K}_U,\dot{K}_1.\dot{K}_I\right\}<1$.

In view of the contraction mapping principle, the operator B has a unique fixed point which is a periodic solution to the neutral system.
Theorem 6.1 is thus proved.

3 Numerical Example

Here we collect all inequalities guaranteeing the existence-uniqueness of the periodic solution:

$$\frac{e^{\mu T_0}(V_0+J_0)}{2}\le\phi_0;\ \frac{e^{\mu T_0}(V_0+J_0)}{2Z_0}\le \mathrm{I}_0;\ \frac{e^{\mu_0}}{\mu\hat{C}_0}\left(\frac{V_0+e^{-\beta}J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right)\le U_{G_0};$$

$$J_0e^{-\beta}+\frac{e^{\mu_0}}{\mu}\frac{Z_0\left(V_0+J_0e^{-\beta}+2U_{G_0}+2U_{E_0}\right)}{\hat{L}_0}\le V_0;\ \frac{e^{\mu_0}}{\mu\hat{C}_1}\left(\frac{V_0e^{-\beta}+J_0}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(1)}\right|U_{G_1}^n e^{(n-1)\mu_0}\right)\le U_{G_1};$$

$$V_0e^{-\beta}+\frac{e^{\mu_0}}{\mu}\frac{Z_0\left(V_0e^{-\beta}+J_0+2U_{G_1}+2U_{E_1}\right)}{\hat{L}_1}\le J_0;$$

$$e^{\mu_0}K_0=\frac{e^{2\mu_0}A_0}{\mu^2}\left[\frac{A_0B_0}{2}\left(\frac{V_0+J_0e^{-\beta}}{2Z_0}+\sum_{n=1}^{m}\left|g_n^{(0)}\right|U_{G_0}^n e^{(n-1)\mu_0}\right)+\sum_{n=1}^{m-1}n\left|g_n^{(0)}\right|U_{G_0}^{n-1}e^{(n-1)\mu_0}+\frac{1}{2Z_0}\right]<1;$$

$$e^{\mu_0} K_U = \frac{e^{\mu 2_0}}{\mu^2}\left[\frac{3Z_0}{\hat{L}_0} + \frac{1}{\hat{L}_{_0}^2}\left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0}\right)\sum_{n=1}^{m}(n+1)n l_n^{(0)}\left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0}\right)^{n-1}\right] < 1;$$

$$e^{\mu_0} K_1 = \frac{e^{2\mu_0} A_1}{\mu^2}\left[\frac{A_1 B_1}{2}\left(\frac{V_0 e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m}\left|g_n^{(1)}\right| U_{G_1}{}^{n} e^{(n-1)\mu_0}\right) + \sum_{n=1}^{m} n\left|g_n^{(1)}\right| U_{G_1}^{n-1} e^{(n-1)\mu_0} + \frac{1}{2Z_0}\right] < 1;$$

$$e^{\mu_0} K_I = \frac{e^{2\mu_0}}{\mu^2}\left[\frac{3Z_0}{\hat{L}_1} + \frac{1}{\hat{L}_{_2}^2}\left(\frac{V_0 e^{-\beta} + J_0}{2} + U_{G_1} + U_{E_1}\right)\sum_{n=1}^{m}(n+1)n l_n^{(1)}\left(\frac{V_0 e^{-\beta} + J_0}{2Z_0}\right)^{n-1}\right] < 1;$$

$$\dot{K}_0 = \left(1 + \frac{e^{\mu_0} - 1}{\mu_0}\right)\frac{A_0}{\mu}\left[\frac{A_0 B_0}{2}\left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0} + \sum_{n=1}^{m}\left|g_n^{(0)}\right| U_{G_0}^{n} e^{(n-1)\mu_0}\right) + \sum_{n=1}^{m-1} n\left|g_n^{(0)}\right| U_{G_0}^{n-1} + \frac{1}{2Z_0}\right] < 1;$$

$$\dot{K}_U = \left(1 + \frac{e^{\mu_0} - 1}{\mu_0}\right)\frac{1}{\mu}\left[\frac{3Z_0}{\hat{L}_0} + \frac{e^{\mu_0}}{\hat{L}_{_0}^2}\left(\frac{V_0 + J_0 e^{-\beta}}{2} + U_{G_0} + U_{E_0}\right)\sum_{n=1}^{m}(n+1)n l_n^{(0)}\left(\frac{V_0 + J_0 e^{-\beta}}{2Z_0}\right)^{n-1}\right] < 1;$$

$$\dot{K}_1 = \left(1 + \frac{e^{\mu_0} - 1}{\mu_0}\right)\frac{A_1}{\mu}\left[\frac{A_1 B_1}{2}\left(e^{\mu_0}\frac{V_0 e^{-\beta} + J_0}{2Z_0} + \sum_{n=1}^{m}\left|g_n^{(1)}\right|\left(U_{G_1}\right)^n e^{n\mu_0}\right) + \sum_{n=1}^{m} n\left|g_n^{(1)}\right| U_{G_1}^{n-1} e^{(n-1)\mu_0} + \frac{1}{2Z_0}\right] < 1;$$

$$\dot{K}_I = \left(1 + \frac{e^{\mu_0} - 1}{\mu_0}\right)\frac{1}{\mu}\left[\frac{3Z_0}{\hat{L}_1} + \frac{e^{\mu_0}}{\hat{L}_{_2}^2}\left(\frac{V_0 e^{-\beta} + J_0}{2} + U_{G_1} + U_{E_1}\right)\sum_{n=1}^{m}(n+1)n l_n^{(1)}\left(\frac{V_0 e^{-\beta} + J_0}{2Z_0}\right)^{n-1}\right] < 1.$$

For a transmission line with length $\Lambda = 1m, \quad L = 0{,}45\,\mu H/m,\ C = 80\ pF/m$,

$v = 1/\sqrt{LC} = 1/\left(6.10^{-9}\right) = 1{,}66.10^{8}$; $Z_0 = \sqrt{L/C} = 75\Omega$. Then, $T = \Lambda\sqrt{LC} = 6.10^{-9}\sec$.

Let us check the propagation of millimeter waves $\lambda_0 = (1/6)10^{-3}m$. We have
$f_0 = 1/\left(\lambda_0\sqrt{LC}\right) = 10^{12}\ Hz \Rightarrow T_0 = 1/f_0 = 10^{-12}$.

We choose $\mu = 10^{12}$, then $\mu T_0 = \mu_0 = 1$ and the resistive elements with the following $V - I$ characteristics: $R_0(i) = R_1(i) = 0{,}028i - 0{,}125i^3$ i.e. $g_1 = 0{,}028, g_2 = 0,\ g_3 = 0{,}125$ and inductive elements with $L_0(i) = L_1(i) = 3i - (1/12)i^3$. Then

$$\overline{L}_0(i) = i\left(dL_0(i)/di\right) + L_0(i) = i(3 - (1/4)i^2) + 3i - (1/12)i^3 = 6i - (1/3)i^3.$$

If we choose $I_0 = 1$, we obtains $6i - (1/3)i^3 > 6 - (1/3) = 17/3$ and consequently $\frac{1}{\hat{L}_0} = \frac{3}{17}$.

Let us take $C_0(u) = c_0/\sqrt{1 - (u/\Phi_0)} = c_0\sqrt{\Phi_0}/\sqrt{\Phi_0 - u}$, where $h = 2$, $c_0 = 50pF = 5.10^{-11}F$ and $\Phi_0 = 0{,}4\ V \Rightarrow \phi_0 < 0{,}4$.

For $\phi_0 = 0,2$ we have

$$C_0(u) = c_0 / \sqrt{1-(u/\Phi_0)} = c_0\sqrt{\Phi_0} / \sqrt{\Phi_0 - u} \geq C_0(-\phi_0) = c_0\sqrt{\Phi_0} / \sqrt{\Phi_0 + V_0} \Rightarrow$$

$$\Rightarrow \hat{C}_0 = 5.10^{-11}\sqrt{0,4} / \sqrt{0,4+0,2} \Rightarrow \hat{C}_0 = 8,2.10^{-11}, \ \mu\hat{C}_0 = 10^{12} 8,2.10^{-11} = 82 .$$

$$A_0 = A_1 = \frac{\sqrt{\Phi_0 + \phi_0}}{c_0\phi_0\sqrt{\Phi_0}} = \frac{\sqrt{0,4+0,2}}{5.10^{-11}.0,2\sqrt{0,4}} \approx \sqrt{1,5}\ .10^{11},$$

$$B_0 = B_1 = \frac{5.10^{-11}\sqrt{0,4}\,(0,8+0,2)}{2\sqrt{0,2^5}} \approx 88,4.10^{-11},\ A_0B_0 = \sqrt{1,5}\ .10^{11}.\frac{250.10^{-11}\sqrt{2}}{4} \approx 108,25 .$$

Then, the above inequalities become for $h = 2$, $V_0 = 0,001$; $J_0 = 0,001$; $U_{G_p} = 0,1$; $U_{E_p} = 0,1$

$$(p = 0,1):\ e\frac{0,002}{2} \leq 0,2; \quad e\frac{0,002}{2.75} \leq 1;\ \frac{e}{82}\left(10^{-3}\frac{1+e^{-\beta}}{150} + 0,028.10^{-1} + 0,125.10^{-3}\right) \leq 0,1;$$

$$J_0e^{-\beta} + \frac{e}{10^{12}}\frac{75\left(V_0 + J_0e^{-\beta} + 0,2 + 0,2\right)3}{17} \leq V_0;\ \frac{e}{82}\left(10^{-3}\frac{e^{-\beta}+1}{150} + 0,028.10^{-1} + 0,125.10^{-3}\right) \leq 0,1;$$

$$V_0e^{-\beta} + \frac{e}{10^{12}}\frac{75\left(V_0e^{-\beta} + J_0 + 0,2 + 0,2\right)3}{17} \leq J_0 .$$

We omit the next four inequalities because $e^{\mu 0}K_0$, $e^{\mu 0}K_U$, $e^{\mu 0}K_1$, $e^{\mu 0}K_I$ are of order $\frac{1}{\mu^2}$. Then

$$\dot{K}_0 = \dot{K}_1 = e\frac{\sqrt{1,5}}{10}\left[\frac{250\sqrt{3}}{8}\left(10^{-3}\frac{1+e^{-\beta}}{150} + 0,028 10^{-1} + 0,125 10^{-3}e^2\right) + 0,028 + 3.0,125 10^{-2} + \frac{1}{150}\right] < 1;$$

$$\dot{K}_U = \dot{K}_I = \frac{3e}{17.10^{12}}\left[225 + \frac{3e}{17}\left(10^{-3}\frac{1+e^{-\beta}}{2} + 0,2\right)\left(12 + 12.\frac{1}{3}10^{-6}\left(\frac{1+e^{-\beta}}{150}\right)^2\right)\right] < 1;$$

or $\dot{K}_0 \approx 0,08 < 1$; $\dot{K}_U = \dot{K}_I \approx \frac{113,1}{10^{12}} < 1$. Consequently, $K \approx 0,08 < 1$.

4 Conclusions

We have considered transmission lines neglecting the losses. This makes it possible to find conditions for the existence and uniqueness of periodic regimes. This natural physical fact is confirmed by the mathematical method we apply.

In contrast to the previous configurations (considered in [10]-[16]), we analyse lossless transmission lines terminated by *L*-loads connected in series to parallel connected GC-loads. The derivation of the boundary conditions shows that we are not able to obtain only two equations for the two unknown functions – the voltage and the current. That is why, we reduce the boundary conditions to a system of four equations for four unknown functions adding some transitional voltages (in [15] transitional currents). The introduced operator acting on suitable function spaces has a fixed point which is a periodic solution to the neutral system obtained.

We apply the contractive fixed point theory (cf. [17]). By extended Bielecki metrics, we overcome the difficulties caused by polynomial and transcendental nonlinearities.

The numerical example demonstrates a frame of applicability of the theory exposed (for instance, to the design of circuits) and shows that the method could be applied checking a system of simple inequalities between the basic specific parameter of the lines and loads. We obtain an explicit approximated solution with a prescribed accuracy.

Competing Interests

Author has declared that no competing interests exist.

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