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Rationality Proof of Newton's Method for Finding Quadratic Trinomial Factors of Univariate Integer Coefficient Polynomials

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The method of finding quadratic trinomial factors for univariate integer coefficient polynomials, proposed by the famous mathematician Isaac Newton in his mathematical monograph *Arithmetica Universalis*, is novel and concise, and has attracted the attention of mathematicians such as Leibniz and Bernoulli. However, no proof of this method has been given so far. This paper provides an in-depth analysis of this method and proves it with mathematical reasoning.Therefore, Newton's method of finding quadratic factors for univariate integer coefficient polynomials is reasonable, validate, and universal.

Keywords: Newton; Algebra; polynomial; quadratic trinomial factor, Arithmetica Universalis.

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1 Introduction

Isaac Newton (1643-1727) was one of the greatest mathematicians in history and made important contributions to algebra [1-6]. He not only applied algebra to physics to solve various problems but also conducted an in-depth and systematical study of algebraic methods [7-13]. In 1707, Newton published a monograph on algebra called *Arithmetica Universalis*, which comprises plentiful knowledge on algebra, such as basic algebraic operations, factorization of polynomials, simplification of radicals, and the method of solving monomial equations, etc. [7,14,15,16] In this book, Newton proposed a method for finding the quadratic trinomial factors of polynomials, which is not only easy to perform but also applicable to all univariate polynomials with integer coefficients. Newton's method had attracted the attention of the famous mathematician Gottfried Wilhelm Leibniz (1646-1716) at that time. Actually, Leibniz did not comprehend this method very well [17-19]. In 1708, Leibniz wrote to Johann Bernoulli (1667-1748) for help, but unfortunately, Bernoulli did not give a clear explanation either [20]. The proof of Newton's method for finding quadratic trinomial factors of a polynomial undoubtedly contributes remarkably to the further understanding and study of Newton's mathematical ideas and methods and contributes to a exhaustive perception of Newtonian scholarship. However, no explicit proof of Newton's method has been given so far. Therefore, this paper attempts to prove Newton's method of finding quadratic factors of univariate integer coefficient polynomials.

2 Newton's Method for Finding Quadratic Trinomial Factors of Univariate Integer Coefficient Polynomials

Newton's Method is introduced in the monograph *Arithmetica Universalis* as follows:

In the first step, the terms of the polynomial are arranged according to the powers of the letters in them, from highest to lowest;

In the second step, the letter of the polynomial is replaced with a series of descending integers with 0 in the middle (the number should be more), such as 3, 2, 0, -1, -2, successively, and obtain the corresponding values;

In the third step, the obtained values followed by their factors are listed, both positive and negative factors;

In the fourth step, the numbers substituted are squared and multiplied by a factor of the highest coefficient of the original polynomial, respectively, then the products are listed;

In the fifth step, the products above subtract all the corresponding factors obtained in the previous second step, then we can get the differences and list them;

In the sixth step, we pick one difference in each line to make them an arithmetic progression;

In the seventh step, the tolerance of the above arithmetic progression is regarded as the first-order coefficient of the quadratic trinomial factor, the opposite of the number in the arithmetic progression corresponding to the substitution number 0 as the constant term, a factor of the highest term coefficient of the original polynomial (i.e., the factor used in the fourth step) as the coefficient of the third term. The quadratic polynomial constructed is a possible quadratic trinomial factor of the original polynomial;

In the eighth step, the real quadratic trinomial factors are determined by the polynomial division.

For example, there is a problem which is to find all the quadratic trinomial factors of the polynomial $x^4 - x^3 - 5xx + 12x - 6$.

Because it has been arranged from high to low according to the degree of the letter x, we start from the second step directly.

The numbers 3, 2, 1, 0, -1, -2 are used to replace x in the polynomial respectively, and we attain six numbers 39, 6, 1, -6, -21, -26.

These six numbers and their factors accordingly are listed(for simplicity, their negative factors are not listed), as shown in Fig. 1.

Then we multiply the factor 1 of the leading coefficient 1 by the square of the six numbers just now to obtain 9, 4, 1, 0, 1, 4, and list them.

These six square numbers subtract the factors in the corresponding row of the front face. The resulting difference is listed to the right of each square number.

$$\begin{array}{r|r|l|l|l|c}
3 & 39 & 1\cdot 3\cdot 13\cdot 39 & 9 & -30\cdot -4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 22\cdot 48 & -4\cdot 6 \\
2 & 6 & 1\cdot 2\cdot 3\cdot 6 & 4 & -2\cdot 1\cdot 2\cdot 3\cdot 5\cdot 6\cdot 7\cdot 10\cdot & -2\cdot 3 \\
1 & 1 & 1 & 1 & 0\cdot 2\cdot & 0\cdot 0 \\
0 & 6 & 1\cdot 2\cdot 3\cdot 6 & 0 & -6\cdot -3\cdot -2\cdot -1\cdot 1\cdot 2\cdot 3\cdot 6 & 2\cdot -3 \\
-1 & 21 & 1\cdot 3\cdot 7\cdot 21 & 1 & -20\cdot -6\cdot -2\cdot 0\cdot 2\cdot 4\cdot 8\cdot 22 & 4\cdot -6 \\
-2 & 26 & 1\cdot 2\cdot 13\cdot 26 & 4 & -22\cdot -9\cdot 2\cdot 3\cdot 5\cdot 6\cdot 17\cdot 30 & 6\cdot -9
\end{array}$$

Fig. 1. Illustration of the first example

After observing the differences, a number from each row is selected to make them an arithmetic progression. With this approach, two arithmetic progressions can be found, namely -4, -2,0,2,4,6 and 6,3,0, -3, -6, -9.

Because the tolerance of the first arithmetic progression is 2 and the resulting number corresponding to the substitution number 0 is 2, and the first term coefficient of the original polynomial is 1, then the quadratic trinomial x^2+2x-2 can be obtained.

Similarly, because the tolerance of the second arithmetic progression is -3, the resulting number corresponding to the substitution number 0 is -3, and the first term coefficient of the original polynomial is 1, a quadratic trinomial x^2-3x+3 can be obtained.

In the following, let the original polynomial try to be divided by two quadratic polynomials, then we find that it is divisible for both the quadratic polynomials, so they are both factors of the original polynomial.
Another example is to find the quadratic trinomial factors of the polynomial $3y^5-6y^4+y^3-8yy-14y+14$.

The numbers 3 , 2 , 1 , 0 , -1 , -2 are used to replace x in the polynomial respectively, and we attain six numbers 170, 38, 10, 14, 10, 190. These six numbers and their factors accordingly are listed(for simplicity, their negative factors and that of 170 and 190 are not listed), as shown in Fig. 2.

The numbers 3 , 2 , 1 , 0 , -1 , -2 are used to replace x in the polynomial respectively, and then multiply them with a factor 3 of the leading coefficient 3 and we attain six numbers 27, 12, 3, 0, 3, 12.

These six numbers subtract the factors in the corresponding row of the front face. The resulting differences are listed to the right of each square number, as shown in Fig. 2.

After observing the differences, a number from each row from top to bottom is picked to make them an arithmetic progression. With this approach, two arithmetic progressions can be found, namely-7,-7,......-7 and 17,11,......-7,-13.

Since the tolerance of the first arithmetic progression is 0, the number corresponding to the substitution number 0 is -7, and the leading coefficient of the original polynomial is 3, we can find the quadratic trinomial $3y^2+7$. Similarly, since the tolerance of the second arithmetic progression differences is -6, the number corresponding to the substitution number 0 is -1, and the leading coefficient of the original polynomial is 3, the quadratic trinomial $3y^2-6y+1$ can be obtained.

Whether the original polynomial is divisible for the above two quadratic trinomials decides whether they are true factors. After examining it shows that $3y^2+7$ is divisible for y^3-2y^2-2y+2, whereas $3y^2-6y+1$ is not, which indicates that only $3y^2+7$ is the factor of original polynomial y^3-2y^2-2y+2.

$$\begin{array}{r|r|l|r|l|r}
3 & 170 & & 27 & & -7\cdot 17 \\
2 & 38 & 1\cdot 2\cdot 19\cdot 38 & 12 & -16\cdot -7\cdot 10\cdot 11\cdot 13\cdot 14\cdot 31\cdot 50\cdot & -7\cdot 11 \\
1 & 10 & 1\cdot 2\cdot 5\cdot 10 & 3 & -7\cdot 2\cdot 1\cdot 2\cdot 4\cdot 5\cdot 8\cdot 13\cdot & -7\cdot 5 \\
0 & 14 & 1\cdot 2\cdot 7\cdot 14 & 0 & -14\cdot -7\cdot -2\cdot -1\cdot 1\cdot 2\cdot 7\cdot 14\cdot & -7\cdot -1 \\
-1 & 10 & 1\cdot 2\cdot 5\cdot 10 & 3 & -7\cdot -2\cdot 1\cdot 2\cdot 4\cdot 5\cdot 8\cdot 13\cdot & -7\cdot -7 \\
-2 & 190 & & 12 & & -7\cdot -13
\end{array}$$

Fig. 2. Illustration of the second example

3 Modern Mathematical Representation of Newton's Method

The description of the above steps in modern mathematical language is:

Let there be an integer coefficient polynomial $F_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ for which the following transformations are made:

(1) An arithmetic progression with tolerance 1 centered at 0 is proposed, such as $p, p-1, p-2, \cdots 1, 0, -1, \cdots -p(p>0)$;

(2) The value of each number in the above arithmetic progression is taken into the polynomial $F_n(x)$ to get $F_n(i)$, where $-p\le i\le p, i\in Z$;

(3) All the integer factors of each $F_n(i)$ are found to make i sets $\{F_n(i)_j \mid j\in Z\}$;

(4) A new progression whose each term is the square of the above arithmetic progression is constructed as $p^2, (p-1)^2, (p-2)^2, \cdots 1^2, 0^2, (-1)^2, \cdots (-p)^2$;

(5) One factor A of the leading coefficient a_n is picked to multiply each term of the obtained progression in the next step to get a new progression as $A\cdot p^2, A\cdot(p-1)^2, A\cdot(p-2)^2, \cdots A\cdot 1^2, A\cdot 0^2, A\cdot(-1)^2, \cdots A\cdot(-p)^2$;

(6) We calculate all the $A\cdot i^2-F_n(i)_j \quad (-p\le i\le p, i\in Z)$ to make i sets $\{G(i)_j=A\cdot i^2-F_n(i)_j \mid j\in Z\}$;

(7) A number $m_{i,k}$ （$1\le k\le j$） from each set $\{G(i)_j=A\cdot i^2-F_n(i)_j \mid j\in Z\}$ is picked to make them be an arithmetic progression with *i* numbers: $m_{p,k_1}, m_{p-1,k_2}, m_{p-3,k_3} \cdots m_{-p,k_{2p+1}}$;

(8) The tolerance of the above arithmetic progression is denoted as B;

(9) The center number m_{0,k_0} is remarked, and a quadratic trinomial can be constructed as (Ax^2+Bx-m_{0,k_0}) which is a potential quadratic factor of the original polynomial;

(10) Let the fifth step to the ninth step repeat to find all the potential quadratic factors of the original polynomial;

(11) All the potential quadratic factors should be examined to be determined whether they are true quadratic trinomial factors.

4 Proof of Newton's Method

Lemma 1: Let (px^2+qx+r) be a quadratic factor of the integer coefficient polynomial $F_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, then $p\,|\,a_n$, $r\,|\,a_0$.

Proof: Because (px^2+qx+r) is a quadratic factor of the integer coefficient polynomial $F_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, the polynomial can be rewritten as $F_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=(px^2+qx+r)(b_{n-2}x^{n-2}+\cdots+b_0)$.

So it is obviously that $a_n=pb_{n-2}$, $a_0=rb_0$, then we will find $p\,|\,a_n$, $r\,|\,a_0$.

Lemma 2: There is an integer coefficient polynomial $F_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, and it has a quadratic trinomial factor (px^2+qx+r), then all the numbers of the arithmetic progression with a tolerance of 1 centered on 0 as $p,p-1,p-2,\cdots1,0,-1,\cdots-p$ is brought into the polynomial respectively, and the integer factors of resulting number $F_n(i)$ ($-p\le i\le p, i\in Z$) are calculated to form a set $\{G(i)_j=p\cdot i^2-F_n(i)_j\mid j\in Z\}$. Finally, a number of each set $\{G(i)_j=p\cdot i^2-F_n(i)_j\mid j\in Z\}$ is selected and represented as $m_{i,k}$ ($1\le k\le j$). So at least one of the progressions consisting of $m_{i,k}$ ($1\le k\le j$) is an arithmetic progression whose tolerance is *q*.

Proof: Because (px^2+qx+r) is a quadratic factor of the integer coefficient polynomial $F_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, the polynomial can be rewritten as $F_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=(px^2+qx+r)(b_{n-2}x^{n-2}+\cdots+b_0)$.

So after all the numbers of the arithmetic progression with a tolerance of 1 centered on 0 viz $p,p-1,p-2,\cdots1,0,-1,\cdots-p$ are taken into the original polynomial, we will find that $(p\cdot i^2+q\cdot i+r)\,|\,F_n(i)$, $-p\le i\le p, i\in Z$.

Let all the integer factors $F_n(i)_j$ of $F_n(i)$ constitute a set $\{F_n(i)_j\mid j\in Z\}$, then we will get $(p\cdot i^2+q\cdot i+r)\in\{F_n(i)_j\mid j\in Z\}$ or $(-q\cdot i-r)\in\{G(i)_j=p\cdot i^2-F_n(i)_j\mid j\in Z\}$.

It can be obtained easily from above that among all progressions consisted of the numbers $m_{i,k}$ ($1\le k\le j$) from $\{G(i)_j=p\cdot i^2-F_n(i)_j\mid j\in Z\}$ listed in accordance with i from the largest to the smallest, we can always find an arithmetic progression whose tolerance is *q*.

With the help of two lemmas proved above, Newton's method for finding quadratic trinomial factors of univariate integer coefficient polynomials can be demonstrated successfully as follows.

As can be seen from Newton's process of finding factors, the method can be divided into three steps. The first is to find an arithmetic progression and find its tolerance; the second step is to construct a quadratic trinomial and find the potential factors; and the third step is to determine the true factors by implementing division.

For the first step, it follows from Lemma 2 that in the presence of a quadratic trinomial factor of a polynomial with integer coefficients, there is necessarily an arithmetic progression, and its tolerance must be the coefficient of the leading coefficient of this quadratic trinomial.

The second step of Newton's method is to construct the coefficients of a quadratic trinomial factor whose highest degree term is chosen to be a factor of the leading coefficient of the original polynomial, which, by Lemma 1, must be the leading coefficient of a potential quadratic trinomial factor of the original polynomial.

And in the second step, the constant term of a quadratic trinomial factor is defined as the opposite number of one of the factors of the value obtained by bringing 0 to the original polynomial, which by Lemma 1 must be the constant term of the potential quadratic trinomial factor.

5 Conclusion

Based on the above analysis, therefore, all the potential quadratic trinomial factors of the original polynomial can be found after the second step of Newton's method.

The third step of Newton's method obviously enables the identification of the true quadratic trinomial factors from the potential quadratic trinomial factors and is therefore reasonable.

In conclusion, Newton's method of finding quadratic trinomial factors is proved to be reasonable.

Competing Interests

Author has declared that no competing interests exist.

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