



## Error Analysis of Meshfree Approximation in Nonlinear Black-Scholes Model

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### Authors' contributions

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

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## Abstract

The transaction cost model of Guy Barles and Halil Mete Soner is incorporated into the standard Black Scholes Equation. The resulting model is solved by a numerical method, called, the meshfree approximation using radial basis function. The errors produced by the scheme are discussed and presented in diagrams and tables.

*Keywords: Black-Scholes; radial basis function; error analysis; differential equations.*

## 1 Introduction

The Black-Scholes option pricing model is the basic platform of current option pricing theory according to Black and Scholes [1].

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During [2] discussed that some of the principles used to formulate the Black-Scholes model proves to be unrealistic in the real world. Ever since, there have been several authors who have formulated the Black-Scholes model that take into account interest rate or transaction costs. This new formulation of the model leads to a nonlinear Black-Scholes model for the option price. However, analytical solutions do not exist in many cases, and hence have to be approximated by numerical technique.

Traditional schemes such as the finite difference, finite volume method and finite element are initially assign on meshes of set of data points. This particular mesh has every single node having a stable number of previously defined neighbours. This link existing between the neighbours are then used to construct mathematical equations such as the derivatives.

In mathematical modelling such as option pricing problems, the material being simulated can move around and thereby destroy the connectivity of the mesh. This introduces errors in the simulation. Meshfree method rectifies the problems caused by the traditional mesh based methods such as the difference method. It does not require a mesh to connect its nodes or data points in the simulation domain.

In this research article, we review the standard Black-Scholes model with transaction cost and solve the resulting nonlinear model with meshfree approximation using radial basis function (*RBF*) for the solution of the option value and discuss the error produce by this scheme.

A couple of authors have recently studied the meshfree methods. Kansa [3] applied multi-quadric RBF for elliptic, parabolic and hyperbolic partial differential equations. His results compared the multiquadrics and that of the finite difference schemes and showed that the multiquadrics is more efficient than the finite difference scheme.

Fedoseyev et al. [3] improved the Kansa multi quadric radial basis function method. Additional set of nodes or data points were added to the adjacent boundary and, correspondingly, additional set of collocation equations were also added to obtained via collocation of the PDE on the boundary. They applied their method to one dimensional and two dimensional linear and nonlinear elliptic partial differential equations.

Sharan et al. [4] used the multiquadric radial basis function and obtained the solution of elliptic partial differential equations with Neumann and Dirichlet boundary conditions. They found the radial basis function and that of the exact solutions to be very good.

Goto et al. [5] did a thorough study on options by applying meshfree approximation using radial basis function.

Belova et al. [6] solved European, Barrier, Asian and American options using meshfree approximation for the numerical solution of the option values.

Onwona-agyeman and Oduro [7] applied radial basis function to obtain the solution of the nonlinear Black-Scholes option pricing equation. Their values were compared to that of the exact solution of the European option pricing equation and varied the transaction cost parameter.

## 2 Materials and Methods

### 2.1 Black-Scholes equation with transaction cost

The Black-Scholes equation, derived by Fischer Black and Myron Scholes in 1973 is given by equation (2.1) below

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \quad (2.1)$$

When transaction cost is incorporated into equation 2.1, the model result in a fully nonlinear Black-Scholes model. According to During [2], he stated the nonlinear equation as:

$$V_t + \frac{1}{2} \hat{\sigma}(S, t, V_s, V_{ss})^2 S^2 V_{ss} + rSV_s - rV = f(S, t, V_s), \quad S > 0, \quad t \in (0, T) \quad (2.2)$$

Where  $r$  is the interest rate,  $\hat{\sigma}$  denotes the nonlinear volatility, and  $f$ , the nonlinear function models the different effects.

## 2.2 The model of Barles and Soner

Barles & Soner [8] did a thorough work on the model of Hodges & Neuberger [9] by applying their utility function approach.

A financial market that consists of a stock and a bond, the price is given by:

$$dS(l) = S(l)[\alpha dl + \sigma dW(l)], \quad l \in [t, T] \quad (2.3)$$

Barles and Soner stated that, the process of bonds owned is  $X(s)$  and the process of shares owned  $Y(s)$ .  $M(s)$  is usually measured in shares from stock to bond and for  $L(s)$ , it is from bond to stock.

If  $\mu \in (0,1)$  represents the transaction cost and initial values  $x$  and  $y$ ,  $s \in [t, T]$ ,

The processes  $X(s)$  and  $Y(s)$  is given by

$$X(s) = x - \int_t^s S(\tau)(1 + \mu)dL(\tau) + \int_t^s S(\tau)(1 - \mu)dM(\tau), \quad s \in [t, T] \quad (2.4)$$

and

$$Y(s) = y + L(s) - M(s), \quad s \in [t, T] \quad (2.5)$$

Given the utility function as

$$U(\xi) = 1 - e^{-\gamma \xi}, \quad \xi \in \mathbb{R} \quad (2.6)$$

then  $\gamma > 0$  is the risk aversion factor of the utility function. If we consider the first and second optimization problems of Boyle & Vorst [10], we have the first value function as the expected utility from final wealth without any option liabilities.

$$V_a(x, y, S(t), t) := \sup_{L(\cdot), M(\cdot)} E [U(X(T) + Y(T)S(T))] \quad (2.7)$$

The second value function can be defined as the expected utility from the final wealth if  $N$  European call options taken over the transfer processes have been sold

$$V_b(x, y, S(t), t) := \sup_{L(\cdot), M(\cdot)} E [U(X(T) + Y(T)S(T) - N((T) - K)^+)] \quad (2.8)$$

The price of every single option is equivalent to the maximal solution  $\Lambda$  of the equation according to Hodges & Neuberger [9]

$$V_a(x + N\Lambda, y, S(t), t) = V_b(x, y, S(t), t)$$

According to Ankudinova [11], the option price  $L$  amounts to the increment of the initial capital at the time  $t$  needed to make out with the option liabilities arising at  $T$ . If  $N$  options with risk aversion factor of  $\gamma$  is sold, then it yields the same price when one option with risk aversion factor  $\gamma N$  is sold by the principle of linearity argument.

We do an asymptotic analysis as  $\gamma N \rightarrow \text{inf}$ .

$$U(\xi) = 1 - e^{-\gamma N \xi}$$

Setting

$$\gamma N = 1/\varepsilon$$

Yields

$$U_\varepsilon(\xi) = 1 - e^{-\xi/\varepsilon} \quad \xi \in \mathbb{R}$$

The optimization problem leads to

$$V_a(x, y, S(t), t) = 1 - \inf_{L(\cdot)M(\cdot)} E [e^{-1/\varepsilon(X(T) + Y(T)S(T))}]$$

$$V_b(x, y, S(t), t) = 1 - \inf_{L(\cdot)M(\cdot)} E [e^{-1/\varepsilon(X(T) + Y(T)S(T) - (S(T) - K)^+)}]$$

Now set,  $z_a$  and  $z_b : \mathbb{R} \times (0, \infty) \times (0, T) \rightarrow \mathbb{R}$  by

$$V_a(x, y, S(t), t) = 1 - e^{-\frac{1}{\varepsilon(x+yS(t)-z_a(y, S(t), t))}}$$

$$V_b(x, y, S(t), t) = 1 - e^{-1/\varepsilon(x+yS(t)-z_b(y, S(t), t))}$$

It yields

$$z_a(y, S(t), T) = 0 \text{ and } z_b(y, S(t), T) = (S(T) - K)^+$$

The option price can then be given as

$$\Lambda(x, y, S(t), t, \frac{1}{\varepsilon}, 1) = z_b(y, S(t), t) - z_a(y, S(t), t)$$

The values of the functions  $V_a$  and  $V_b$  are the unique solutions of the equation

$$\min \left\{ -V_t + \frac{1}{2} \sigma^2 S^2 V_{ss} - rSV_s, -V_y + S(1 + \mu)V_x, -V_y + S(1 - \mu)V_x \right\} = 0 \tag{2.9}$$

We do a dynamic programming for  $z_a$  and  $z_b$ , which are not dependent on  $x$ . If the proportional transaction cost be  $\mu = a\sqrt{\varepsilon}$  for some constant  $a > 0$ , then as  $\varepsilon \rightarrow 0, \mu \rightarrow 0, z_a \rightarrow 0$  and  $z_b \rightarrow V$ ,

$$V_t + \frac{1}{2} \hat{\sigma}^2 S^2 V_{ss} + rSV_s - rV = 0 \tag{2.10}$$

with the nonlinear part given as

$$\hat{\sigma}^2 = \sigma^2(1 + \Psi(e^{r(T-t)}a^2S^2V_{ss})) \quad (2.11)$$

Where  $a = \mu/\sqrt{\varepsilon}$  and  $\Psi(x)$  will be the solution of the nonlinear ODE

$$\Psi'(x) = \frac{\Psi(x)+1}{2\sqrt{x\Psi(x)-x}} \quad x \neq 0 \quad (2.12)$$

$$\text{with initial condition} \quad \Psi(0) = 0 \quad (2.13)$$

From equations (2.12) and (2.13), it can be analyzed that

$$\lim_{n \rightarrow \infty} \frac{\Psi(x)}{x} = 1 \quad \text{and} \quad \lim_{n \rightarrow -\infty} \frac{\Psi(x)}{x} = -1 \quad (2.14)$$

For large argument of the function  $\Psi(\cdot)$ , it can be treated as identity, hence, we have

$$\hat{\sigma}^2 = \sigma^2(1 + (e^{r(T-t)}a^2S^2V_{ss})) \quad (2.15)$$

### 2.3 Application of radial basis function

We solve the nonlinear Black-Scholes model with meshfree approximation using radial basis function.

#### 2.3.1 Discretization

The value of the option  $V$ ; is approximated using

$$V(S, t) \approx \sum_{j=1}^N b_j(t)\phi(S, S_j) \quad (2.16)$$

where  $\phi(S, S_j)$  are the radial basis functions and  $b_j$  are some unknown coefficients. Multiquadric radial basis function is used for this problem

$$\phi(S, S_j) = \sqrt{c^2 + \|S - S_j\|^2} \quad (2.17)$$

Where  $S_j$  is the asset price at the collocation point  $j$ ,  $\|S - S_j\|$  represents the radial distance of each of the  $N$  scattered data points  $S_j$ . The shape parameter is represented by  $c$ .

The nonlinear Black-Scholes model holds for the option price  $V(S, t)$  with asset price  $S$  at time  $t$

$$\frac{\partial V(S, t)}{\partial t} + \frac{1}{2}\hat{\sigma}^2S^2\frac{\partial^2 V(S, t)}{\partial S^2} + rS\frac{\partial V(S, t)}{\partial S} - rV = 0 \quad (2.18)$$

$$\hat{\sigma}^2 = \sigma^2(1 + (e^{r(T-t)}a^2S^2V_{ss})) \quad (2.19)$$

with equation 2.19 as the nonlinear term.

Setting the initial condition as

$$V(S, t) = \begin{cases} \max(S(T) - K, 0) & \text{for call option} \\ \max(K - S(T), 0) & \text{for put option} \end{cases}$$

with  $T$  as the time of maturity and  $K$ , the strike price.

Differentiating Equation 2.16 leads to

$$\frac{\partial V(S, t)}{\partial t} = \sum_{j=1}^N \frac{db_j(t)}{dt} \phi(S, S_j) \quad (2.20)$$

$$\frac{\partial V(S, t)}{\partial S} = \sum_{j=1}^N b_j \frac{\partial \phi(S, S_j)}{\partial S} \quad (2.21)$$

$$\frac{\partial^2 V(S, t)}{\partial S^2} = \sum_{j=1}^N b_j \frac{\partial^2 \phi(S, S_j)}{\partial S^2} \quad (2.22)$$

The derivatives of equation 2.17 are also given by

$$\frac{\partial \phi}{\partial S} = \frac{S - S_j}{\sqrt{(S - S_j)^2 + c^2}} \quad (2.23)$$

$$\frac{\partial^2 \phi}{\partial S^2} = \frac{1}{\sqrt{(S - S_j)^2 + c^2}} - \frac{(S - S_j)^2}{\sqrt{((S - S_j)^2 + c^2)^3}} \quad (2.24)$$

Simplifying and applying the theta method of the Crank-Nicholson scheme results in

$$HV(S, t + \Delta t) = GV(S, t) \quad (2.25)$$

Where

$$G = 1 - \theta \Delta t \left( \frac{1}{2} S^2 \frac{\partial^2}{\partial S^2} \left( 1 + \left[ e^{r(T-t)} a^2 S^2 \frac{\partial^2}{\partial S^2} \right] \right) + rS \frac{\partial}{\partial S} - r \right)$$

$$H = 1 + (1 - \theta) \Delta t \left( \frac{1}{2} S^2 \frac{\partial^2}{\partial S^2} \left( 1 + \left[ e^{r(T-t)} a^2 S^2 \frac{\partial^2}{\partial S^2} \right] \right) + rS \frac{\partial}{\partial S} - r \right)$$

The option price is estimated with RBF, and is governed by equation 2.16. If equation 2.16 is substituted into equation 2.25, we have

$$Hb_j(t + \Delta t) \phi \| S - S_j \| = Gb_j(t) \phi \| S - S_j \| \quad (2.26)$$

We solve equation 2.26 for  $b_j$  by using

$$V(S, t) \approx \sum_{j=1}^N b_j(t) \phi(S, S_j)$$

The accuracy of the numerical solution can be assessed by calculating the absolute percentage error  $E_r$  as

$$E_r(S, t) = \frac{|V_{rbf}(S,t) - V_{exact}(S,t)|}{V_{exact}(S,t)} \times 100$$

Where  $V_{rbf}(S, t)$  is the option value obtained by the RBF and  $V_{exact}(S, t)$  is the option value of the European option pricing equation.

### 3 Numerical Results and Discussion

The parameter values provided in (Onwona-Agyeman and Oduro, [7]) and the algorithm suggested in (Goto et al. [5]) for evaluating the option price,  $V(S, t)$  is used.

MATLAB R2018b software is used for all the values and diagrams obtained for the European option with and without transaction cost.

Maximum asset price ( $S_{max}$ ) = 30, Number of asset data points ( $N$ ) = **121**, Transaction Cost parameter ( $\alpha$ ) = **0.005**, Shape parameter ( $c$ ) = **0.01**, Number of time steps ( $\Delta t$ ) = **0.05**, Expiration date ( $T$ ) = **0.05**, Exercise price ( $K$ ) = **10**, Crank-Nicholson method ( $\theta$ ) = **0.5**, Volatility ( $\sigma$ ) = **0.2**, Risk free interest rate ( $r$ ) = **0.05**.

From Table 2, the exact value of the option and the option value with transaction cost increases with increasing asset price,  $S$ . For a maximum asset price of 30.0, the option vales for both the exact value and that of the transaction cost are 20.2469 and 20.2743 respectively.

**Table 1. The comparison of exact value and option value with transaction cost**

Asset price, $S$	Exact value	Option value with transaction cost
0.0	0.0000	0.0000
2.0	0.0000	0.0000
4.0	0.0000	0.0000
6.0	0.0000	0.0000
8.0	0.0456	0.0561
10.0	0.6888	0.6983
12.0	2.2952	2.3104
14.0	4.2496	4.2681
16.0	6.2470	6.2781
18.0	8.2469	8.2500
20.0	10.2469	10.2960
22.0	12.2469	12.2877
24.0	14.2469	14.3010
26.0	16.2469	16.2899
28.0	18.2469	18.2654
30.0	20.2469	20.2743

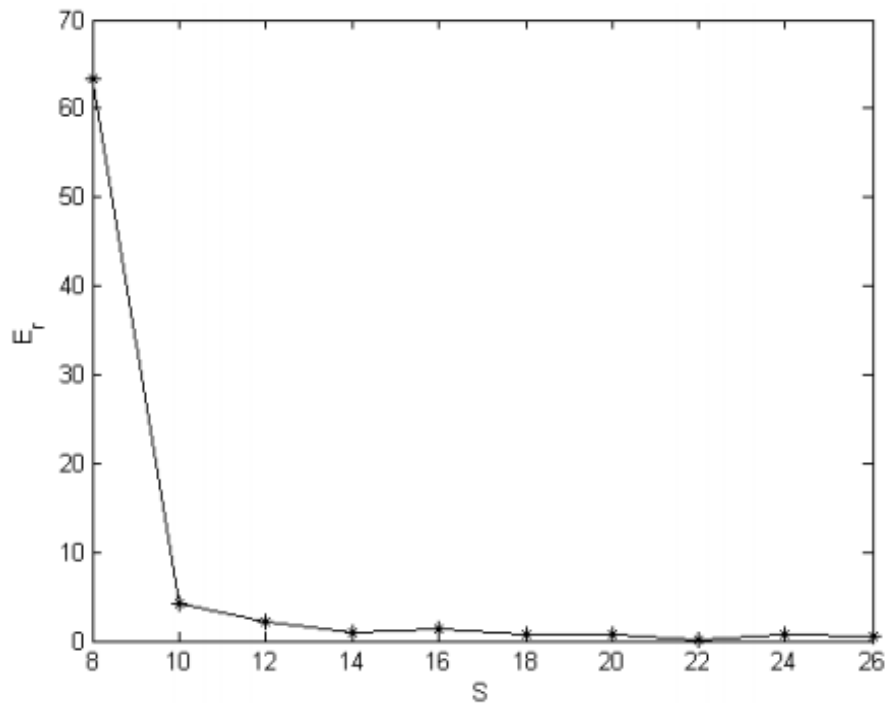
Because the RBFs can be differentiated, We can easily obtain the derivatives of the options from the basis functions. The value of the option delta can be calculated from equation 3.76. The values of delta for European options with transaction

cost using RBF method is given in Table 2. From Table 2, the option delta lies between 0 and 1.

The error decreases for bigger values of the asset price  $S$ , in Fig. 1. For an asset price of 10, the RBF approximation error is 5 percent. For an asset price of 26, the RBF approximation error is 45 percent.

**Table 2. Values of option's delta when a = 0:01**

Asset price; S	Exact values of the option's delta	Numerical values of the option's delta
8.0	0.1243	0.1241
10.0	0.6643	0.6642
12.0	0.9567	0.9564
14.0	0.9975	0.9972
16.0	0.9999	0.9996
18.0	1.0000	0.9998
20.0	1.0000	0.9999



**Fig. 1. Plot of percentage error,  $E_r$  versus asset price, S**

## 4 Conclusion

We reviewed option pricing and radial basis function in general in this paper. The RBF was then applied to solve the Barles and Soner option pricing model. This led to a system of differential equations which were solved by the Crank- Nicholson Scheme. The numerical results of the values of the option were presented and discussed. A comparison between the influence of transaction costs on the value of the option to that of the analytical Black-Scholes option price was discussed. The percentage errors were computed and was found that, an asset of 26, has an RBF approximation error is 0.45 percent. This means that, the numerical method used to approximate the option value gives a minimum error of less than 5% for increasing asset price.

## Competing Interests

Authors have declared that no competing interests exist.



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