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# Primaries Oblateness Effects on the Collinear Libration Points in the Restricted-three Body Problem

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***Authors' contributions***

*This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.*

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## Abstract

In this work, the canonical Hamiltonian form of the restricted three- body problem including the effects of primaries oblateness is presented. Moreover, the collinear libration points are obtained. In addition to this, the relation between position of libration points and variation in (mass ration , oblateness coefficients A1 and A2) is studied. The results obtained are a good agreement with Perdios [1] & Singh [2]. The Poincare surface section PSS is used to illustrate the stability of motion around each of the collinear libration points. A numerical application on the real system Earth-Moon is presented.

*Keywords:* *Restricted three body problem; oblateness effects; Poincare surface of section; stability motion; libration points.*

## 1 Introduction

A special case of general three body problem is the restricted three body problem (RTBP) which play an important role in celestial mechanics. The equations of motion, in general case are non–linear in nature and

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it is more difficult to obtain analytic solutions for the general problem. Hence some restrictions were put to overcome this problem as the circular and elliptical RTBP. The restricted three-body problem with oblate primaries has also attracted the interest of many Researches [3-6]. Perdios [1] investigated the combined influence of the oblateness and radiation pressures of the primaries on collinear points moreover calculated Lyapunov planar and 3D family of periodic orbits around these points. Markello [6] obtained zero velocity and libration points for Hill's problem with the effect of oblateness. Abdul Raheem [7] found the periodic orbits around the triangular libration points under the effects of the centrifugal and the Coriolis forces together with solar radiation and oblateness of the two primaries. Singh [8] studied the nonlinear stability of the triangular points when both primaries are oblate spheroids. Ibrahim [9] obtained the Lissajous orbits and the phase spaces around collinear points under the effect of oblateness. The stability of a test particle moves about the equilibrium points in the circular restricted three-body problem is studied with both primaries oblateness and radiation together with P–R drag [10]. Analysing LSS in the Earth-Moon system, exploring dynamical structures that are available within a multi-body gravitational environment are investigated [11]. The Laplace transformations are used to study the effect of mass variation on the stability of libration points of the restricted three-body problem [12]. Equations of motion for the perturbed circular restricted three-body problem are regularized in canonical variables in a moving coordinate system, two different L-matrices of the fourth order are used in the regularization [13].

The perturbed mean motion n of the primaries is given by $n^2 = 1 + \frac{3}{2}(\mathrm{A1} + \mathrm{A2})$ , where $Ai = \frac{r_{ei}^2 - r_{pi}^2}{5\,R^2}$ is the oblateness coefficient of $m_1$ and $m_2$ having the equatorial and polar radii as $r_{ei}$ and $r_{pi}$, respectively and R is separation between the primaries. Then the effective potential equation with oblateness of two primaries is given by

$$V(x,y,z) = \frac{n^2}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)A1}{2r_1^3} + \frac{\mu\,A2}{2r_2^3} \tag{1}$$

Where

$$r_1 = \sqrt{(\mathrm{x}-\mu)^2 + y^2} \tag{2.1}$$

$$r_2 = \sqrt{(\mathrm{x}+1-\mu)^2 + y^2} \tag{2.2}$$

## 2 The Hamiltonian System of R3BP with Oblateness

The Hamiltonian of the restricted three body problem with oblateness can be written as

$$\mathcal{H} = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + yp_x - x\,p_y - \frac{n^2}{2}(x^2+y^2) - \left[\frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)A1}{2r_1^3} + \frac{\mu\,A2}{2r_2^3}\right] \tag{3}$$

It is clear that Equation [5] has more two terms than the previous Hamiltonian obtained by many authors, which are $\frac{(1-\mu)A1}{2r_1^3}\,and\,\frac{\mu\,A2}{2r_2^3}$ and the mean motion n depends on the oblateness.

The canonical form is given by

$$\dot{x} = \frac{\partial\mathcal{H}}{\partial p_x} = p_x + y \tag{4.1}$$

$$\dot{y} = \frac{\partial\mathcal{H}}{\partial p_y} = p_y - x \tag{4.2}$$

$$\dot{z} = \frac{\partial\mathcal{H}}{\partial p_z} = p_z \tag{4.3}$$

$$\dot{p}_x = -\frac{\partial\mathcal{H}}{\partial x} = -p_y - n^2x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu\,(x-\mu+1)}{r_2^3} - \frac{3A1\,(1-\mu)\,(x-\mu)}{2r_1^5} - \frac{3\,A2\,\mu\,(x-\mu+1)}{2\,r_2^5} \tag{4.4}$$

$$\dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = p_x - n^2 y - \frac{(1-\mu).\ y}{r_1^3} - \frac{\mu\ y}{r_2^3} - \frac{3\,A1\,(1-\mu)y}{2\,r_1^5} - \frac{3\,A2\,\mu\ y}{2\,r_2^5} \tag{4.5}$$

$$\dot{p}_z = -\frac{\partial \mathcal{H}}{\partial z} = -\frac{(1-\mu).\ z}{r_1^3} - \frac{\mu\, z}{r_2^3} - \frac{3\,A1\,(1-\mu)\,.\ z}{2\,r_1^5} - \frac{3\,A2\,\mu\, z}{r_2^3} \tag{4.6}$$

Equations (4) represent the equation of motion of the third body under the effect of gravitational forces and the Oblateness of two primaries.

# 3 Location of the Libration Points with Effect of Oblateness

To obtain the location of libration points, put

$p_x = p_y = p_z = \dot{p}_x = \dot{p_y} = \dot{p_z} = 0$, then eqns. (4.4), (4.5) and (4.6) will be

$$-n^2 x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu\ (x-\mu+1)}{r_2^3} - \frac{3A1\,(1-\mu)(x-\mu)}{2r_1^5} - \frac{3\,A2\,\mu\ (x-\mu+1)}{2\,r_2^5} = 0 \tag{5.1}$$

$$-n^2 y - \frac{(1-\mu).\ y}{r_1^3} - \frac{\mu\ y}{r_2^3} - \frac{3\,A1\,(1-\mu)y}{2\,r_1^5} - \frac{3\,A2\,\mu\ y}{2\,r_2^5} = 0 \tag{5.2}$$

$$-\frac{(1-\mu).\ z}{r_1^3} - \frac{\mu\, z}{r_2^3} - \frac{3\,A1\,(1-\mu)\,.\ z}{2\,r_1^5} - \frac{3\,A2\,\mu\, z}{r_2^3} = 0 \tag{5.3}$$

The collinear points can be determined from equation (5.1).

$$n^2 x = \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu\ (x+\mu-1)}{r_2^3} - \frac{3A1\,(1-\mu)(x+\mu)}{2r_1^5} - \frac{3\,A2\,\mu\ (x+\mu-1)}{2\,r_2^5} \tag{6}$$

The locations of the collinear points are

$$x_1 = \mu - 1 - X1, \qquad x_2 = \mu - 1 + X2\,, \qquad x_3 = \mu + X3.$$

where$X1, X2$ and X3 satisfy ninth degree polynomials while in Ibrahim [14] it was seventh degree when one of primary has oblate spheroid. By expanding equation (6) up to sixth power in x, then the first point is obtained by solving the next equation which gives only one real root, this real root is the location of libration point

$$\mathrm{X1}^9(2+3\mathrm{A1}+3\mathrm{A2}) + \mathrm{X1}^8(10+15\mathrm{A1}+15\mathrm{A2}-2\mu-3\mathrm{A1}\mu-3\mathrm{A2}\mu) + \mathrm{X1}^7(20+30\mathrm{A1}+30\mathrm{A2}-8\mu-12\mathrm{A1}\mu-12\mathrm{A2}\mu+\mathrm{X1}618+30\mathrm{A1}+30\mathrm{A2}-8\mu-18\mathrm{A1}\mu-18\mathrm{A2}\mu+\mathrm{X1}56+15\mathrm{A1}+15\mathrm{A2}+4\mu-12\mathrm{A1}\mu-12\mathrm{A2}\mu+\mathrm{X1}46\mathrm{A1}+3\mathrm{A2}+12\mu-6\mathrm{A1}\mu+\mathrm{X1}38\mu+12\mathrm{A2}\mu+\mathrm{X1}22\mu+18\mathrm{A2}\mu+\mathrm{X1}12\mathrm{A2}\mu+3\mathrm{A2}\mu = 0 \tag{7}$$

Similarly for the second and third points

$$(2+3\mathrm{A1}+3\mathrm{A2})\mathrm{X2}^9 + (-10-15\mathrm{A1}-15\mathrm{A2}+2\mu+3\mathrm{A1}\mu+3\mathrm{A2}\mu)\mathrm{X2}^8 + (20+30\mathrm{A1}+30\mathrm{A2}-8\mu-12\mathrm{A1}\mu-12\mathrm{A2}\mu\mathrm{X2}7+-18-30\mathrm{A1}-30\mathrm{A2}+8\mu+18\mathrm{A1}\mu+18\mathrm{A2}\mu\mathrm{X2}6+6+15\mathrm{A1}+15\mathrm{A2}+4\mu-12\mathrm{A1}\mu-12\mathrm{A2}\mu\mathrm{X2}5+-6\mathrm{A1}-3\mathrm{A2}-12\mu+6\mathrm{A1}\mu\mathrm{X2}4+8\mu+12\mathrm{A2}\mu\mathrm{X2}3+-2\mu-18\mathrm{A2}\mu\mathrm{X2}2+12\mathrm{A2}x\mu\mathrm{X2}-3\mathrm{A2}\mu = 0 \tag{8}$$

$$(2+3\mathrm{A1}+3\mathrm{A2})\mathrm{X3}^9 + (8+12\mathrm{A1}+12\mathrm{A2}+2\mu+3\mathrm{A1}\mu+3\mathrm{A2}\mu)\mathrm{X3}^8 + (12+18\mathrm{A1}+18\mathrm{A2}+8\mu+12\mathrm{A1}\mu+12\mathrm{A2}\mu\mathrm{X3}7+10+12\mathrm{A1}+12\mathrm{A2}+8\mu+18\mathrm{A1}\mu+18\mathrm{A2}\mu\mathrm{X3}6+10+3\mathrm{A1}+3\mathrm{A2}-4\mu+12\mathrm{A1}\mu+12\mathrm{A2}\mu\mathrm{X3}5+12-3\mathrm{A1}-12\mu+6\mathrm{A1}\mu\mathrm{X3}4+8-12\mathrm{A1}-8\mu+12\mathrm{A1}\mu\mathrm{X3}3+2-18\mathrm{A1}-2\mu+18\mathrm{A1}\mu\mathrm{X3}2+-12\mathrm{A1}+12\mathrm{A1}\mu\mathrm{X3}-3\mathrm{A1}+3\mathrm{A1}\mu=0 \tag{9}$$

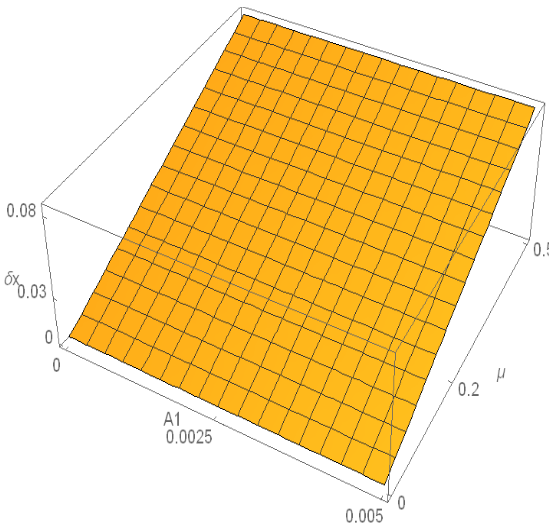


**Fig. 1.a Variation of L1 under the effect of A1**

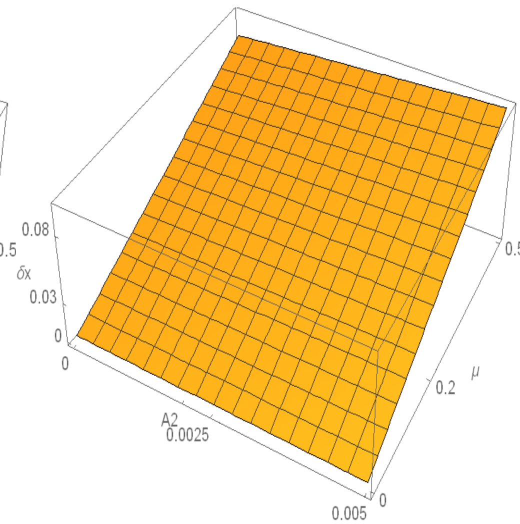


**Fig. 1.b Variation of L1 under the effect of A2**

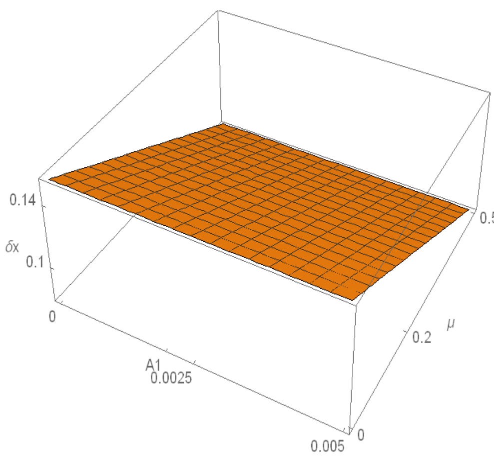


**Fig. 2.a Variation of L2 under the effect of A1**

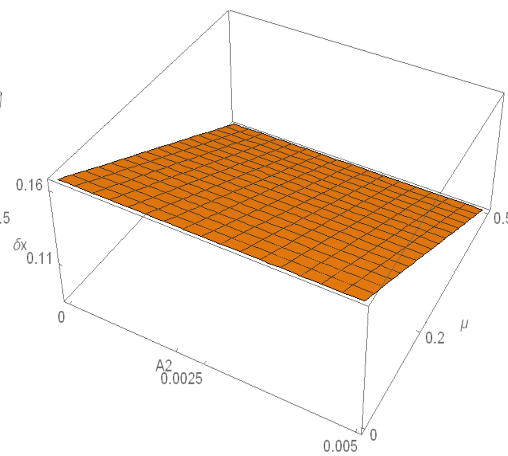


**Fig. 2.b Variation of L2 under the effect of A2**

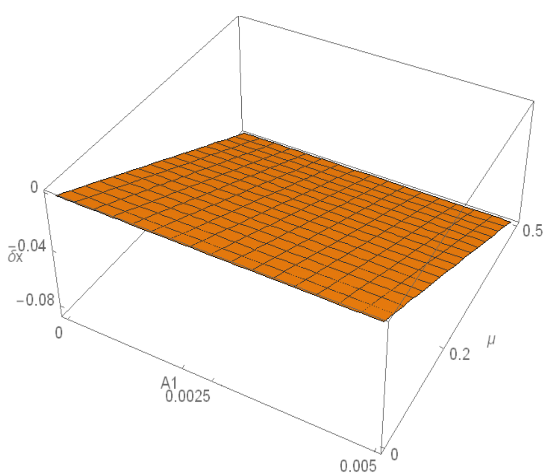


**Fig. 3.a Variation of L3 under the effect of A1**

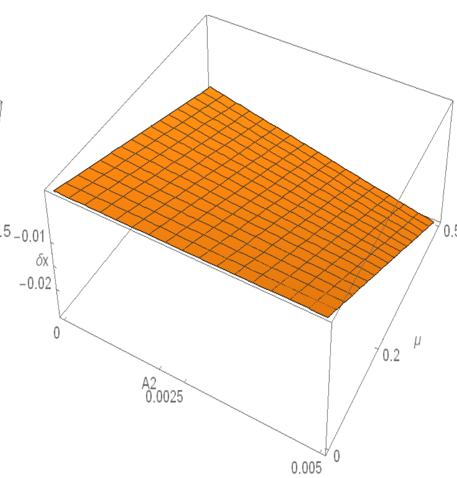


**Fig. 3.b Variation of L3 under the effect of A2**

Figs. 1–3 show surface representations of the variation of collinear libration (L1, L2, L3) under the effects of oblateness for the bigger and smaller primaries where μ ∈ (0,.5], A1 and A2 ∈ [0, 0.005]. Where fig 1 and fig 3 represent the increase in variation of L1 and L3 when μ, A1 and A2 increase contrary with fig2 the decrease in variation of L2 when μ, A1 and A2 increase. this coincide with [3,4].

# 4 Poincare Surfaces of Section

In the restricted three-body problem, Poincare surface of Section (PSS) is a powerful technique for studying stability of orbits which enables finding stable periodic and quasiperiodic around the two primaries [15]. The four-dimensional phase space $(x, y, \dot{x}, \dot{y})$ is used to obtain (PSS) of the infinitesimal body at any instant. This is a good tool to study the stability of nature system which enables determine the regular or chaotic nature of the trajectory. Figures (4) and (5) show Poincare surface section, when $A1 = A2 = 0$ and $A1 \neq 0, A2 \neq 0$ for Earth-Moon system. The numerous islands can be observed which means that the behavior of the trajectory is likely to be regular, where the curves shrink down to a point, it represents a periodic orbit. To obtain this PSS an initial condition of x varies from 0.6 to 0.8 with δx = 0.001are used. So that, the canonical equations of motions (4) are integrated truncated up to 1000 steps using Runge–Kutta fixed step sized method [16].

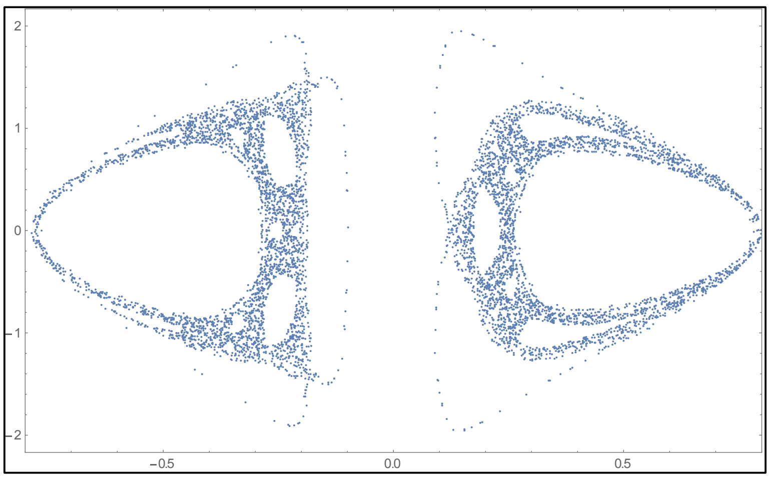


**Fig. 4. Poincare surface sections for earth-moon system A1= A2 = 0**

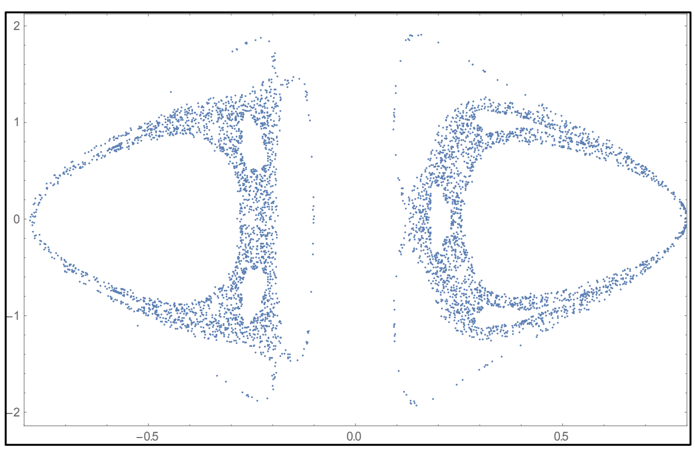


**Fig. 5. Poincare surface sections for earth-moon system A1≠A2 ≠0**

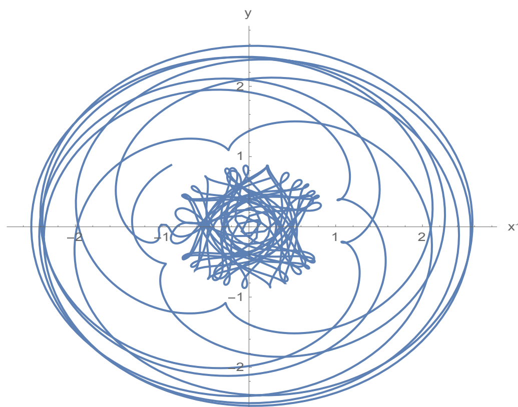


**Fig. 6.a Time from 0 to 200**

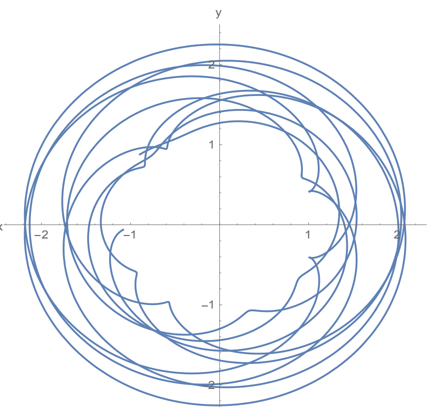


**Fig. 6.b Time from 200 to 324**

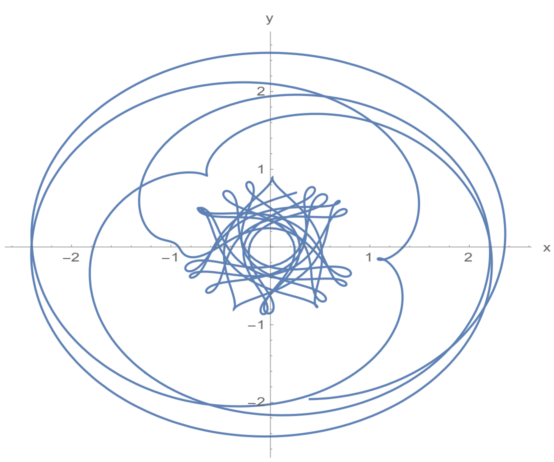


**Fig. 7.a t from 0 to 100**

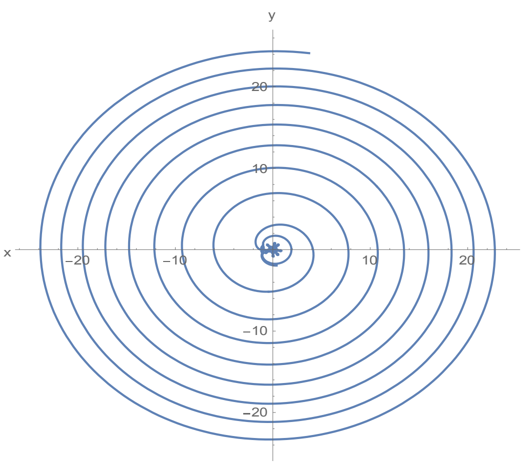


**Fig. 7.b t from 100 to 200**

Fig. 6, and Fig. 7 represent the orbits of the third body. Figs.6 displays the orbit in presence of oblateness effect while Fig.7 displays orbit in absence of oblateness effect. The orbit of the infinitesimal body represents in first frame when $0 \leq t \leq 200$ whereas second frame when $200 \leq t \leq 340$. In the second frame, in the case of absence of oblateness effect, the orbit as spiral shape. However, with effect of oblateness, orbit becomes regular when $0 \leq t \leq 200$ which is shown in third frame while fourth frame shows the orbit when $100 \leq t \leq 200$.

## 5 Conclusion

This study related to the motion of the third body under the effects of the oblateness of the two primaries. It is found that the positions of L2 decreased under the effects of oblateness while the positions of L1 and L3 are increased under this effect. A PSS shows the stability of nature system which are a regular motion in the intervals where x belongs to ].15, 0.8[and ]-0.2,-.8[, while $\dot{x}$ belongs to ]-1.8,1.8[, and a chaotic motion are investigated in the intervals x belongs to]-.15, 0.15[while $\dot{x}$ belongs to ]-1.8,-2[ and ]1.8,2[ for the system without oblateness and under the effects of oblateness. These kinds of study are very important for the space missions. In near future the study will truncated under the effects of solar radiation pressure and the oblateness which is very important for the motion of solar sails.

## Competing Interests

Authors have declared that no competing interests exist.

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