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Review Paper

Moore-Penrose inverse of linear operators in Hilbert space

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In this paper, we investigate properties of $T \in C(H)$ with closed range satisfying the operator equations $TT^+T = T$, $T^+TT^+ = T^+$, $(TT^+)^* = TT^+$ and $(T^+T)^* = T^+T$. In particular, we investigate the invertibility of $T \in C(H)$ with closed range where the Moore-Penrose inverse of T turns out to be the usual inverse of T under some classes of operators. We also deduce the Moore-Penrose inverse of a perturbed linear operator $(T + S)$ with closed range where $T = PQ$ such that $P, Q \in C(H)$ has closed ranges and S $\in B(H)$ satisfying some given conditions. The relation between the ranges and null spaces of these operators is also shown.

Key words: Moore-Penrose inverse, perturbed linear operator, invertibility of operators.

INTRODUCTION

The invertibility of linear operators is useful in finding the solution to the operator equation $Tx = y$ where y is a given vector and x an unknown vector. The inverse of T exists and is unique if and only if T is bijective. If $y \notin R(T)$, the operator equation has no solution. Also, if $N(T) \neq \{0\}$, then $Tx = y$ has many solutions. In such cases generalized inverses of T are used. It is known that the generalized inverse of $T \in L(H)$ exist if and only if the range of T is closed. However, there exists a unique generalized inverse called the Moore-Penrose inverse which gives the best approximate solution. That is $x_1 \in D(T)$ such that $\|Tx_1 - y\|$ for all $x \in D(T)$ is of minimum norm where $x_1 = T^+y$ and T^+ is the Moore-Penrose inverse of T.

The Moore-Penrose inverse of operators with closed range has been considered by several authors among them: Moore (1920) and Penrose (1955) came up with operator equations satisfied by the Moore-Penrose inverse. That is the operator T^+ is the solution to the following equations:

$TT^+T = T$, $T^+TT^+ = T^+$, $TT^+ = (TT^+)^*$ and $T^+T = (T^+T)^*$.

Generalized inverses for matrices and linear operators

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Mathematics subject classification: 47B20 have been studied by several scholars among them:

Campbell and Meyer (1991), Roger and Johnson (1985), Rao et al. (1971), James (1978) among others. Drazin (1958), (2012), and (2016) studied Pseudo-inverses in associative rings and semigroups. The author gave a theory for a large class of uniquely-defined outer generalized inverses. Moreover, Drazin (2016) gave a way to define left and right versions of the large class of (b, c)-inverses. Wang et al. (2017) gave some characterizations of the (b, c)-inverse in terms of the direct sum decomposition, the annihilator and the invertible elements. Baksalary and Trenkler (2010) introduced the notion of the core inverse as an option to the group inverse and gave its properties. Mary (2011) studied generalized inverses on semigroups by means of green's relations. The author first defined an inverse along an element and studied its properties. Penrose et al. (1955) described a generalized inverse of a non-singular matrix as a unique solution for some operator equations. Rakic et al. (2014) showed that the core generalized inverses are closely related and also gave several characterizations of these inverses.

The reverse order of Moore-Penrose inverse, $(TS)^+ = S^+T^+$ has been investigated by several authors among them: Israel and Greville (2003), Djordjevic and Dijana (2011). The authors gave conditions which imply $(TS)^+ = S^+T^+$. Brock (1990) gave a characterization of an EP operator in Hilbert spaces. In particular, the author showed equivalent conditions for a bounded linear operator with closed range. Koliha (2000) gave different conditions which imply that an operator is an EP operator. The author specifically gave equations which imply each other for an upper semi-Fredholm operator. That is, for an upper semi-Fredholm operator T, then, $T^+T = TT^+$, $(T^*T)_\pi T = 0$, $T(TT^*)_\pi = 0$ implies the other.

Several authors have discussed results on perturbation of operators in Hilbert spaces and Banach spaces among them: Chen and Xue (1997), Chen et al. (1996), Ding (2003), Ding and Huang (1997), Stewart (1977), Wei (2003), Wei and Chen (2001), Zhou and Wang (2007), Wei and Ding (2001) came up with a detailed formula for the generalized inverse of the perturbed operator under some conditions. Deng and Wei (2010) generalised the result of Wei and Ding (2001) under different conditions. Shani and Sivakumar (2013) discussed rank-one perturbations of closed range operators and obtained the Moore-Penrose inverse of the operators. Kulkarni and Ramesh (2015) gave an equation for Moore-Penrose inverse of a perturbed linear operator as $(T+S)^+ = (I+T^+S)^{-1}T^+$ where $S \in B(H)$ is a perturbation of T and satisfies some given conditions.

They also gave the relation between the range of the operators. That is, conditions under which the closed range of T implies closeness of range of T+S. We contribute to this study by showing that the Moore-Penrose inverse of an EP operator can be the usual inverse of the operator under the given conditions. Under perturbation of linear operators, we give the Moore-Penrose inverse of $(T+S)$ where $T = PQ \in C(H)$ with closed ranges and $S \in B(H)$ under some given conditions distinct from the ones used by Kulkarni and Ramesh (2015). We come up with corollaries relating to these theorems and also show that $R(T+S)$ is closed if $R(P)$ is closed.

NOTATIONS AND TERMINOLOGY

We will use H to denote a Hilbert space, $L(H)$ the space of linear operators on H, $C(H)$ the space of linear operators with closed range on H and B(H) the space of bounded linear operators on H. The range of T will be denoted by $R(T)$ and $\overline{R(T)}$ its closure, its null space by $N(T)$ and the orthogonal complement of its null space by $N(T)^\perp$. If $T \in L(H)$ has dense range then $\overline{R(T)} = H$.

If two operators T and S commute, then $[T,S] = TS - ST = 0$. We denote the commutants of T by T^c.. $T \in L(H)$ is bounded from below if for a scalar $m > 0$, we have $||Tx|| \geq m||x||$ for all x in H. Given an operator $T \in C(H)$ with closed range, we define the generalized inverse of T as the operator $T' \in L(H)$ Satisfying $T = TT'T$. Also there exists a unique operator T^+ called the Moore-Penrose inverse of T which is a unique operator satisfying the following four conditions:

$TT^+T = T$, $T^+TT^+ = T^+$,
$TT^+ = (TT^+)^*$ and $T^+T = (T^+T)^*$ such that TT^+ is an orthogonal projection on $R(T^+)$ and $N(T^+) = R(T)^\perp$,

We use W(T) for numerical range of T. The Drazin inverse of T is a unique element T_D satisfying the following conditions: $TT_D = T_DT, T_DTT_D = T_D, T^{p+1}T_D = T^p$ for some non-negative integer $p \geq 1$. An operator T is said to have a spectral idempotent T_π at 0 if $TT_\pi = T_\pi T$ is quasinilpotent, $T_\pi = T_\pi^2$ and $T + T_\pi$ is invertible.
An operator T is:

(1) simply polar if $TT_\pi = 0$ where $T_\pi = I - T_D T$.
(2) upper semi-Fredholm if its range is closed and either its kernel or codimension of R(T) is finite.
(3) An EP operator if $R(T) = R(T^*)$.
(4) R- quasi- EP operator if $[T, TT^+] = 0$.

(5) L-quasi EP operator if $[T, T^{+}T] = 0$.
(6) Partial isometry if $T = TT^{*}T$ or $T^{+} = T^{*}$.
(7) Quasinormal if $[TT^{*}, T] = 0$.
(8) Quasinilpotent if its spectrum contains only the zero scalar.

MAIN RESULTS

Theorem 1

Let $T \in C(H)$ with closed range and T' its generalized inverse. If:

(i) T is one to one, then T'T=I.
(ii) $\overline{R(T)} = H$, then $TT' = I$.

Proof

(i) The generalized inverse of T satisfies $T = TT'T$. This implies $T - TT'T = 0$ and $T(I - T'T) = 0$. Since T is one-to-one, then $I - T'T = 0$. Thus $T'T = I$.
(ii) If $T' \in B(H)$ is the generalized inverse of $T \in C(H)$ with closed range, then $R(T) = R(TT'T) \subseteq R(TT') \subseteq R(T)$. Thus, R(T)= $R(TT')$. If T has a closed range that is dense in H, then T its onto. This implies that $R(T) = R(TT') = H$. Thus, $T = TT'T$ implying $T - TT'T = 0$ and $(I - TT')T = 0$. Since $\overline{R(T)} = H$, then T≠ 0 and hence, $I - TT' = 0$. Implying. $TT' = I$.

We note that since $R(T) = R(TT') = H$, $T': R(T) = H \to D(T)$, then $TT': H \to H$ is defined on H. It is known that an operator T has a right inverse if R(T)=H.

Corollary 1

Let $T \in C(H)$ with closed range and T^{+} be the Moore-Penrose inverse of T. If T is a quasiaffinity, then $T^{+} = T^{-1}$.

Proof

T being a quasiaffinity implies that it has a dense range and is one-to-one. From Theorem 1, $T'T = TT' = I$. Since T^{+} is a unique generalized inverse of T, then $T^{+} = T^{-1}$.

Remark 1

In Corollary 1, we have deduced that $T^{+} = T^{-1}$ for the case of a quasiaffinity. In Theorem 2, we relax the condition of quasiaffinity in Corollary 1 to operator with either dense range or injective under the proviso that T is an EP operator.

Theorem 2

If $T \in C(H)$ with closed range is an EP operator and T^{+} its Moore-Penrose inverse, then $T^{+} = T^{-1}$ in each of these cases:

(i) T is one-to-one.
(ii) $\overline{R(T)} = H$.

Proof

The Moore-Penrose inverse of T satisfies $T = TT^{+}T$. If T is an EP operator, then $T^{+}T = TT^{+}$. This implies, $T = TT^{+}T = TTT^{+}$ and $T = TT^{+}T = T^{+}TT$. Thus, $T(I - T^{+}T) = 0$ and $T(I - TT^{+}) = 0$.
Also, $(I - TT^{+})T = 0$ and $(I - T^{+}T)T = 0$.
If T is injective then $I - TT^{+} = 0$ and $I - T^{+}T = 0$.
Thus $T^{+}T = TT^{+} = I$, implying $T^{+} = T^{-1}$. Also, if T has a closed range and $\overline{R(T)} = H$, then it is onto implying $T^{+}T = I$ and $TT^{+} = I$ where TT^{+} is defined on H. Hence $T^{+} = T^{-1}$.

In Corollary A, Brock (1990) gave a characterization of EP operator as follows.

Corollary A (Brock (1990))

The following statements are equivalent for $T \in B(H)$ with a closed range.

(i) $T^{+}T = TT^{+}$.
(ii) $H = N(T) \oplus R(T)$.
(iii) $N(T) = N(T^{*})$.
(iv) $T^{*} = PT$ for some invertible operator P in H.

Corollary 2

If $T \in C(H)$ has a closed range and it is either one-to-one or has a dense range, then $T^{+} = T^{-1}$ in each of the cases listed:

(i) $H = N(T) \oplus R(T)$.
(ii) $N(T) = N(T^{*})$.
(iii) $T^{*} = PT$ for some invertible operator P in H.

Proof

From Corollary A above, each of conditions (i) – (iii) implies T is an EP operator. Thus, the proof of Theorem 2 carries through.

Corollary B (Koliha (2000))

The following statements are equivalent for T, an upper semi-Fredholm operator on H.

(i) $T^{+}T = TT^{+}$.
(ii) $(T^{*}T)_{\pi}T = 0$.
(iii) $T(TT^{*})_{\pi} = 0$.

Remark 2

In Corollary 3, we use the result of Koliha (2000) in Corollary B which uses another special inverse called the Drazin inverse and the spectral idempotent of T at 0 to show that the Moore-Penrose inverse of an operator is the same as inverse of an operator under some conditions.

Corollary 3

If $T \in C(H)$ is an upper semi-Fredholm operator on H with a closed range and either $(T^{*}T)_{\pi}T=0$ or $T(TT^{*})_{\pi} = 0$. Then $T^{+} = T^{-1}$ in each of the cases:

(i) T is one to one.
(ii) $\overline{R(T)} = H$.

Proof

From Corollary B, T is an EP operator hence the proof of Theorem 2 carries through.

Theorem C (Wong ((1986)**)**

$T \in B(H)$ is an EP operator if and only if its Moore-Penrose inverse is a polynomial of T provided that H has a finite dimension.

Remark 3

From Theorem C we note that every subspace of a finite dimensional space is closed hence if $T \in B(H)$ where H is a finite dimensional space, then it has a closed range and if its range is dense in H, then it's surjective. Again, if T is one to one, then it follows that T is invertible. Hence, $T^{-1} = T^{+}$.

Theorem D (Khalagai and Sheth (1987))

If $T, S \in B(H)$ satisfy $[S, T^{2}] = 0$, then $[S, T] = 0$ in each of the cases listed.

(i) $T^{c} = \{T^{2m}\}^{c}$ for some positive integer m.
(ii) T is normal and either $\sigma(\mathrm{Re}\,T) \cap \sigma(-\mathrm{Re}\,T) = \emptyset$ or $\sigma(\mathrm{Im}\,T) \cap \sigma(-\mathrm{Im}\,T) = \emptyset$.
(iii) T is normal and either $0 \notin W(\mathrm{Re}\,T)$ or $0 \notin W(\mathrm{Im}\,T)$.

Corollary 4

If $T \in C(H)$ has a closed range such that $[T^{+}, T^{2}] = 0$ and it's either one-to-one or $\overline{R(T)} = H$, then $T^{+} = T^{-1}$ in each of the statements listed.

(i) $T^{c} = \{T^{2m}\}^{c}$ for some positive integer m.
(ii) T is normal and either $\sigma(\mathrm{Re}\,T) \cap \sigma(-\mathrm{Re}\,T) = \emptyset$ or $\sigma(\mathrm{Im}\,T) \cap \sigma(-\mathrm{Im}\,T) = \emptyset$.
(iii) T is normal and either $0 \notin W(\mathrm{Re}\,T)$ or $0 \notin W(\mathrm{Im}\,T)$.

Proof

From Theorem D, each of the conditions (i) – (iii) implies that T is an EP operator. Since T is either injective or has closed range which is dense in H, then the proof of Theorem 2 carries through.

Lemma E (Anderson [2011])

If $T \in C(H)$ is bounded from below, then:

(i) $N(T) = \{0\}$ and R(T) is closed.
(ii) $N(T^{n}) = \{0\}$ and R(T) is closed.

Remark 4

From Lemma E, it is worth noting that if an operator is bounded from below then the operator is one-to-one and has a closed range. Also, if it is normal with closed range, it is an EP operator.

Corollary 5

If $T \in B(H)$ is normal and bounded from below, then $T^{+} = T^{-1}$.

Proof

By Lemma E, T is one to one with a closed range and if normal, then it is an EP operator. Thus, the proof of Theorem 2 carries through.

Proposition F (Israel and Greville (2003))

The statements that follow implies the other for $T \in B(H)$ with dense domain.

(i) $R(T)$ is closed
(ii) $R(T^*)$ is closed
(iii) T^+ is bounded
(iv) $R(T^*T)$ is closed
(v) $R(TT^*)$ is closed

Corollary 6

If $T \in B(H)$ is a densely defined normal operator with $\overline{R(T)} = H$, then $T^+ = T^{-1}$ if either $R(T^*)$ or $R(T^*T)$ or $R(TT^*)$ are closed or T^+ is bounded.

Proof

The aforementioned conditions imply that T has a closed range and if its normal, then it is an EP operator. Again, if $\overline{R(T)} = H$, then the proof of Theorem 2 carries through.

Theorem 3

If $T \in B(H)$ is R- quasi –EP operator bounded from below, then $T^+ = T^{-1}$.

Proof

If T is R- quasi- EP operator then from definition T commutes with TT^+ and $T = TT^+T = TTT^+$. This implies $T = TT^+T$ and $T = TTT^+$.

Also, $T - TT^+T = 0$ and $T - TTT^+ = 0$. Since T is linear, then $T(I - T^+T) = 0$ and $T(I - TT^+) = 0$. Since T bounded from below by Lemma E, $N(T) = \{0\}$ implying $I - T^+T = 0$ and $I - TT^+ = 0$. Consequently, $T^+T = I$ and $TT^+ = I$. Hence, $T^+ = T^{-1}$.

Corollary 7

If $T \in L(H)$ is a partial isometry which is bounded from below and T^* is quasinormal, then T is unitary.

Proof

From definition of a partial isometry $T^* = T^+$. If T^* is quasinormal, we have $TTT^* = TT^*T = T$. Substituting T^* for T^+ we have $T = TT^+T = TTT^+$. Thus, the proof of Theorem 3 carries through.

Theorem 4

If $T \in C(H)$ is L- quasi – EP operator with a closed and dense range, then $T^+ = T^{-1}$.

Proof

If T is L - quasi- EP operator then from definition T commutes with T^+T. Thus $(TT^+)T = (T^+T)T$. Since T^+ exists, then $T = TT^+T$. Thus $T = TT^+T = T^+TT$. This implies $T = TT^+T$ and $T = T^+TT$. Hence $T - TT^+T = 0$ and $T - T^+TT = 0$. Also, $(I - TT^+)T = 0$ and $(I - T^+T)T = 0$. Since $\overline{R(T)} = H$, then $I - T^+T = 0$ as well as $I - TT^+ = 0$. Thus, $T^+T = I$ and $TT^+ = I$ implying $T^+ = T^{-1}$. If T has a closed range that is dense in H, then $R(T) = R(TT^+) = H$.

In view of Theorem 4, we extend the result of Mwanzia et al. (2021) in Corollary G for the case of partial isometries.

Corollary G (Mwanzia et al. (2021))

Let $T \in B(H)$ be partial isometry such that $\overline{R(T)} = H$. If T is quasinormal, then T is unitary.

Corollary 8

Let $T \in C(H)$ be a partial isometry with closed range. If T is quasinormal with $\overline{R(T)} = H$, then $T^+ = T^{-1}$.

Proof

From definition of partial isometry, $T^* = T^+$ and $T^*TT = TT^*T = T$. Substituting T^* for T^+, we have, $T^+TT = TT^+T = T$. Since T has a closed range with $\overline{R(T)} = H$, the proof of Theorem 4 carries through.

Remark 5

In the sequel, we study the relation between the range and null spaces of operators and derive the Moore-Penrose inverse of (T+S) where $T = PQ$ and $PP^{+}S = S$ or $SQ^{+}Q = S|_{D(Q)}$ with $S \in B(H)$ being a perturbation of T. Frigyes and Bela (1955) gave the following result which helps in showing the relation between the ranges of operators in Hilbert space.

Proposition H (Frigyes and Bela (1955))

If $T, P \in B(H)$ are densely defined and $S \in B(H)$, then $ST + SP = S(T + P)$.

Theorem 5

If $T = PQ \in C(H)$ and $S \in B(H)$, where P and Q are densely defined with closed ranges. Then

(i) $R(T) \subseteq R(P)$
(ii) $R(T + S) \subseteq R(P)$ if and only if $PP^{+}S = S$.

Proof

Let $y \in R(T)$ and $x \in D(T)$. This means $y = Tx = PQx$. Thus $PQx = y$ for $Qx \in D(P)$. Thus, $y \in R(P)$ implying $R(T) \subseteq R(P)$. Next, if $R(T + S) \subseteq R(P)$, then PP^{+} is an orthogonal projection onto $R(T + S)$. This means, $T + S = PP^{+}(T + S) = PP^{+}PQ + PP^{+}S$ $= PQ + PP^{+}S = T + PP^{+}S$. That is $PP^{+}S = S$. Conversely, let $PP^{+}S = S$ then $T + S = PQ + PP^{+}S$. By Proposition H above, $T + S = P(Q + P^{+}S)$. Hence, $R(T + S) \subseteq R(P)$.

Remark 6

From results of Theorem 5, $PP^{+}T = T$ and $PP^{+}(T + S) = T + S$ under the said conditions.

Corollary 9

If $T = PQ \in C(H)$ is densely defined and P is bounded from below, then $P^{+}PQ = Q$.

Proof

If $T = PQ$ then by Theorem 5, PP^{+} is a projection on $R(T)$. That is $PP^{+}T = T$. Thus, $PP^{+}PQ = PQ$ implying $PP^{+}PQ - PQ = 0$ and $P(P^{+}PQ - Q) = 0$. Since P is bounded from below then by Lemma E, P has a closed range and $N(P) = \{0\}$. Thus, $P^{+}PQ - Q = 0$ and $P^{+}PQ = Q$. This implies $P^{+}P$ is an orthogonal projection on $R(Q)$.

Proposition I (Kulkarni and Ramesh (2015))

If $T \in C(H)$ be densely defined, then:

(i) $N(T) = R(T^{*})^{\perp} = R(T^{+})^{\perp}$.
(ii) $N(T^{*}) = R(T)^{\perp} = N(T^{+})$.

Lemma J (Israel et al (2003))

If $T \in B(H)$ and $||T|| < 1$, then $(1 - T)$ is bijective.

Corollary 10

Let $T = PQ \in C(H)$ where P and Q are densely defined operators with closed ranges. If Q is surjective and $S \in B(H)$ satisfies $PP^{+}S = S$, $SQ^{+}Q = S|_{D(Q)}$ and $||P^{+}SQ^{+}|| < 1$, then $R(T + S) = R(P)$.

Proof

From Theorem 5,

$$R(T + S) \subseteq R(P) \tag{1}$$

Next, let $y \in R(P)$ for $x \in D(P)$ thus $y = Px$. From Lemma J, if $||P^{+}SQ^{+}|| < 1$, then $(I + P^{+}SQ^{+})^{-1} \in B(H)$.
Since Q is surjective and $(I + P^{+}SQ^{+})^{-1}$ exists, then $(I + P^{+}SQ^{+})Q$ is surjective. Thus, there exist $u \in D(P)$ such that $x = (I + P^{+}SQ^{+})Qu$. Thus:

$$\begin{aligned} y &= Px \\ &= P(1 + P^{+}SQ^{+})Qu \\ &= (P + PP^{+}SQ^{+})Qu \\ &= (PQ + PP^{+}SQ^{+}Q)u \\ &= (PQ + PP^{+}S)u \\ &= (T + S)u \end{aligned}$$

That is $(T + S)u = y$ implying $y \in R(T + S)$. Hence,

$$R(P) \subseteq R(T + S) \tag{2}$$

From (i) and (ii) we have $R(P) = R(T + S)$.

Theorem 6

Let $T = PQ \in C(H)$ where P is densely defined operator and bounded from below. If $S \in B(H)$ satisfies $PP^+S = S, SQ^+Q = S|_{D(Q)}$ and $||P^+SQ^+|| \le 1$, then $N(Q) = N(T + S)$.

Proof

For $x \in N(Q)$, we have $(T + S)x = (PQ + PP^+SQ^+Q)x$
$= P(I + P^+SQ^+)Qx = P(I + P^+SQ^+)0$.
Since $||P^+SQ^+|| \le 1$ then $(I + P^+SQ^+)^{-1}$ is invertible and $P(I + P^+SQ^+)0 = P0$. Since P is bounded from below then by Lemma E above, P is injective and thus $(T + S)x = (PQ + SQ^+Q)x$
$= P(I + P^+SQ^+)Qx = P(I + P^+SQ^+)0 = P0$
$= 0$. This means $x \in N(T + S)$ implying
$N(Q) \subseteq N(T + S) \ldots\ldots\ldots (*)$

Next, let $x \in N(T+S)$, then
$0 = (T+S)x = (PQ + PP^+SQ^+Q)x$
$= P(I + P^+SQ^+)Qx$.
That is $P(I + P^+SQ^+)Qx = 0$. Since P is injective, then $(I + P^+SQ^+)Qx = 0$. Also if $||P^+SQ^+|| \le 1$, then $(I + P^+SQ^+)^{-1}$ exists implying $I + P^+SQ^+$ is injective. Thus, $(I + P^+SQ^+)Qx = 0$ implying $Qx = 0$. Hence, $x \in N(Q)$ implying
$N(T + S) \subseteq N(Q) \ldots\ldots\ldots\ldots (**)$

From $(*)$ and $(**)$, then $N(Q) = N(T + S)$.

Corollary 11

If $T = PQ \in C(H)$ where P is bounded from below with dense domain, then $N(Q) = N(T)$.

Proof

For $x \in N(Q)$ means $Tx = PQx = P0$. Since P is bounded
from below then Lemma E, P is injective implying $P0 = 0$. Hence $Tx = P0 = 0$ implying $Tx = 0$. Hence,

$$N(Q) \subseteq N(T) \qquad \text{(i)}$$

Next, let $x \in N(T)$. This means $PQx = Tx = 0$ implying $PQx = 0$. Since P is injective then $Qx = 0$. That is $x \in N(Q)$ implying:

$$N(T) \subseteq N(Q) \qquad \text{(ii)}$$

From (i) and (ii) we have $N(T) = N(Q)$.

Theorem 7

Let $T = PQ \in C(H)$ where P and Q are densely defined operators with closed ranges. If Q is surjective and $S \in B(H)$ satisfies $SQ^+Q = S|_{D(Q)}$, $PP^+S = S$ and $||P^+SQ^+|| < 1$, then

(i) $R(T + S)$ is closed.
(ii) $(T + S)^+ = Q^+(I + P^+SQ^+)^{-1}P^+$.

Proof

From Corollary 10, $R(T + S) = R(P)$. Thus if $R(P)$ is closed, then $R(T + S)$ is closed and $(T + S)^+$ exist. Also $T + S = (PQ + SQ^+Q) = (P + PP^+SQ^+)Q = P(I + P^+SQ^+)Q$. Thus, $T + S = P(I + P^+SQ^+)Q$. If $||P^+SQ^+|| < 1$ then by Lemma J, $1 + P^+SQ^+$ is bijective and $(1 + P^+SQ^+)^{-1} \in B(H)$. Thus, $(T + S)^+ = Q^+(I + P^+SQ^+)^{-1}P^+$. This equation satisfies the properties of Moore-Penrose as follows:

Let $y \in R(T + S)^+ = R\{Q^+(I + P^+SQ^+)^{-1}P^+\}$. This implies that for $x \in D(P^+)$, we have $Q^+(I + P^+SQ^+)^{-1}P^+x = y$. Multiplying each side by Q from the left we have, $QQ^+(I + P^+SQ^+)^{-1}P^+x = Qy$. Since $QQ^+$is an orthogonal projection onto $R(Q)$, we have $(I + P^+SQ^+)^{-1}P^+x = Qy$ and $P^+x = (I + P^+SQ^+)Qy$. Thus $(I + P^+SQ^+)Qy \in R(P^+)$.
Since P^+P is an orthogonal projection onto $R(P^+)$, we have

$$\begin{aligned}&(T + S)^+(T + S)y\\&= Q^+(I + P^+SQ^+)^{-1}P^+P(I+P^+SQ^+)Qy\\&= Q^+(I + P^+SQ^+)^{-1}(I + P^+SQ^+)Qy\\&= Q^+Qy\end{aligned}$$

Also, $Q^+(I + P^+SQ^+)^{-1}P^+x = y$ implying $y \in R(Q^+)$. Thus $(T + S)^+(T + S)y = Q^+Qy = y$. That is

$$(T + S)^+(T + S)y = P_{\overline{R(T+S)^+}}\, y \ \forall y \in N(T + S)^{\perp} \qquad \text{(i)}$$

Next, we let $y \in R(T+S) = R\{P(I+P^+SQ^+)Q$. This implies $y \in R(P)$ thus, there exist $x \in D(Q)$ such that $P(I+P^+SQ^+)Qx = y$. Multiplying each side from the left by P^+ we have $P^+P(I+P^+SQ^+)Qx = P^+y$. By definition of Moore-Penrose inverse P^+P is an orthogonal projection onto $R(P^+)$ implying $(I+P^+SQ^+)Qx = P^+y$ and $Qx = (I+P^+SQ^+)^{-1}P^+y$.

Thus $(I+P^+SQ^+)^{-1}P^+y \in R(Q)$. Since QQ^+ is an orthogonal projection on $R(Q)$, then we have:

$$\begin{aligned}(T+S)(T+S)^+y &= P(I+P^+SQ^+)QQ^+(I+P^+SQ^+)^{-1}P^+y.\\ &= P(I+P^+SQ^+)(I+P^+SQ^+)^{-1}P^+y.\\ &= PP^+y.\\ &= y.\end{aligned}$$

Thus $(T+S)(T+S)^+y = y$.

This implies:

$$(T+S)(T+S)^+y = P_{\overline{R(T+S)}}\, y = y \ \text{ for all } y \in R(T+S) \qquad \text{(ii)}$$

Lastly, from Proposition H, $(T+S) \in B(H)$ is densely defined operator then

$$N(T+S)^+ = R(T+S)^{\perp} \qquad \text{(iii)}$$

Equations (i), (ii) and (iii) justify the results.

Corollary 12

Let $T = PQ \in C(H)$ where P and Q are densely defined operators with closed ranges. If Q is surjective and $S \in B(H)$ satisfies $SQ^+Q = S|_{D(Q)}$, $PP^+S = S$ and $||P^+SQ^+|| < 1$, then $P^+ = (1+P^+SQ^+)Q(T+S)^+$.

Proof

From Theorem 7, we have $(T+S)^+ = Q^+(I+P^+SQ^+)^{-1}P^+$. Since Q is onto then, $QQ^+ = I$ and multiplying both sides from the left by $(1+P^+SQ^+)Q$ we have, $P^+ = (1+P^+SQ^+)Q(T+S)^+$. Following the steps of Theorem 7, $P^+ = (1+P^+SQ^+)Q(T+S)^+$ satisfies the properties of Moore-Penrose inverse.

Corollary 13

Let $T = PQ \in C(H)$ where P and Q are densely defined operators with closed ranges. If P is bounded from below, Q is surjective and $S \in B(H)$ satisfies $PP^+S = S$, $SQ^+Q = S_{|D(Q)}$ and $||P^+SQ^+|| < 1$ then:

$$Q^+ = (T+S)^+P(I+P^+SQ^+).$$

Proof

From Theorem 7, $(T+S)^+ = Q^+(I+P^+SQ^+)^{-1}P^+$. Since P and $(I+P^+SQ^+)$ are injective then multiplying both sides of the preceding equation by $P(I+P^+SQ^+)$ we have $Q^+ = (T+S)^+P(I+P^+SQ^+)$.

Following the steps of theorem 7, we have:

$Q^+ = (T+S)^+P(I+P^+SQ^+)$ satisfies the properties of Moore- Penrose inverse.

CONFLICT OF INTERESTS

The authors have not declared any conflict of interests.

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