Asian Journal of Physical and Chemical Sciences

6(1): 1-9, 2018; Article no.AJOPACS.41034
ISSN: 2456-7779

Predictor-Corrector Linear Multistep Method for Direct Solution of Initial Value Problems of Second Order Ordinary Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJOPACS/2018/41034
Editor(s):
(1) Sinem Gokturk, Professor, Department of Basic Pharmaceutical Sciences, Faculty of Pharmacy, Marmara University, Turkey.
Reviewers:
(1) E. O. Omole, Joseph Ayo Babalola University, Nigeria.
(2) Adeyeye Oluwaseun, Universiti Utara Malaysia, Malaysia.
Complete Peer review History: http://sciencedomain.org/review-history/24493

Received 10th February 2018
Accepted 18th April 2018
Published 7th May 2018

Original Research Article

ABSTRACT

In this paper, we considered method of interpolation of the approximate solution and collocation of the differential system to generate a continuous linear multistep method. The basic properties of the method was investigated and found to be zero stable, consistent, P-stable and convergent. The method was tested on numerical examples solved by the existing methods, our method was found to performed better in terms of accuracy.

Keywords: Consistent; interpolation; collocation; convergent; off-grid point; ordinary differential equation; predictor-corrector.

2018 Mathematics Subject Classification: 53C25, 83C05, 57N16.

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1 INTRODUCTION

Mathematical modeling of real- life problems usually result into functional equation, for example, Ordinary differential equation and Partial differential equation, Integro and Integral differential equation, Stochastic differential equation and others. Not all ordinary differential equations such as those used to model real life problems can be solved analytically.

In this paper, we consider solving directly second order initial value problems (IVPs) of ordinary differential equations (ODEs) of the form

$$y'' = f(x, y, y'), y(a) = 0, y'(a) = y_0', x\epsilon[a, b] \quad (1.1)$$

Equation (1.1) arises from many physical phenomena in a wide variety of applications especially in engineering such as the motion of rocket or satellite, fluid dynamic, electric circuit and other area of application. Many scholars have worked at solving (1.1) numerically by reducing it to a system of first order equations [1], [2] and others. In spite of the success of this approach, it suffers some setbacks, according to [3] and [4], the setback are; un-economical in term of cost of implementation, increased computational burden and wastage of computer time, increased dimension of the resulting systems of equations to be solved. The method becomes inefficient when the given system of equation to be solved cannot be solved explicitly with respect to the derivative of the highest order. The approach for solving the system of higher order ODEs directly has been suggested by [5], [6], [7] and [8]), according to [4] and [9], continuous linear multistep method have greater advantages over the discrete method, they gives better error estimate and provides a simplified form of coefficient for further analytical work at different points and guarantee easy approximation of solution at all interior points of the integration interval. Among the authors that proposed the linear multistep method are Kayode [5] and [10] to mention few. They developed an implicit linear multistep method which was implemented in predictor corrector mode and adopted Taylor's series expansion to provide the starting value. These authors independently proposed methods of various order of accuracies to proffer solution to problem (1.1) at only grid points. In addition a few authors [11] and [12] have introduced hybrid methods to solving problem (1.1) but with lower accuracies.

In this research work, power series was used as basis function in generating the continuous hybrid linear multistep for the solution of problem (1.1).

2 METHODOLOGY

We consider a power series approximate solution of the form

$$p(x) = \sum_{j=0}^{(c+i)-1} a_j x^j \quad (2.1)$$

where c and i are the number of collocation and interpolation points respectively

The second derivatives of (2) gives

$$y''(x) = \sum_{j=0}^{(c+i)-1} j(j-1)a_j x^{j-2} = f(x, y, y') \quad (2.2)$$

Equation (2.1) and (2.2) are respectively interpolated and collocated at selected grid points and off-grid point to obtain the required methods.

Equation (2.1) was interpolated at one grid point and at one off grid point $x = x_{n+1}$ and x_{n+r}. Equation (2.2) was collocated at four grid points $x = x_{n+i}, i = 0, 1, 2, 3, 4$ and evaluating at the end point i.e $x = x_{n+i}, i = 3$ respectively, gave rise to system of equations which can be express in matrix form

$$AX = B \quad (2.3)$$

$$\begin{bmatrix} 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 \\ 1 & x_{n+r} & x_{n+r}^2 & x_{n+r}^3 & x_{n+r}^4 & {x_{n+r}}^5 & {x_{n+r}}^6 \\ 0 & 0 & 2 & 6\,x_n & 12\,x_n^2 & 20\,x_n^3 & 30\,x_{n+1}^4 \\ 0 & 0 & 2 & 6\,x_{n+1} & 12\,x_{n+1}^2 & 20\,x_{n+1}^3 & 30\,x_{n+1}^4 \\ 0 & 0 & 2 & 6\,x_{n+2} & 12\,x_{n+2}^2 & 20\,x_{n+2}^3 & 30\,x_{n+2}^4 \\ 0 & 0 & 2 & 6\,x_{n+3h} & 12\,x_{n+3}^2 & 20\,x_{n+3}^3 & 30\,x_{n+3}^4 \\ 0 & 0 & 2 & 6\,x_{n+4} & 12\,x_{n+4}^2 & 20\,x_{n+4}^3 & 30\,x_{n+4}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} y_{n+3} \\ y_{n+r} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}$$

This is then solved using Gaussian Elimination method to obtained the parameters a_j's which is then substituted in (2.1) to obtain the continuous system (2.4) after some algebraic simplifications. Using the transformation in [11]

$$t = \frac{x - x_{n+k-1}}{h}$$
$$\frac{dt}{dx} = \frac{1}{h}$$

The coefficients are put as follows:

$$\alpha_3 = \frac{-rh + th + 3\,h}{-rh + 3\,h}$$
$$\alpha_{\frac{7}{2}} = -\frac{th}{-rh + 3\,h}$$
$$\beta_0 = \frac{1}{1440}\left(\frac{45\,th^6 - 147\,th^6 r + 112\,th^6 r - 17\,t^2 h^6 - 2\,trh^6}{h^4}\right)$$
$$\beta_1 = \frac{1}{360}\left(\frac{108\,th^6 - 89\,th^6 r + 24\,t^2 h^6 - 19\,trh^6}{h^4}\right) \tag{2.4}$$
$$\beta_2 = \frac{1}{240}\left(\frac{189\,th^6 - 12\,th^6 r - 41\,t^2 h^6 - 41\,th^6 r + 16\,trh^6}{h^4}\right)$$
$$\beta_3 = \frac{1}{360}\left(\frac{144\,th^6 + 188\,t^2 h^6 + 5\,th^6 r - 13\,trh^6}{h^4}\right)$$
$$\beta_4 = -\frac{1}{1440}\,\frac{27\,th^6 - 12\,th^6 r - 135\,t^2 h^6 + 19\,th^6 r - 12\,trh^6 + 2\,trh^6}{h^4}$$

The first derivative of (2.4) are follows:

$$\alpha'_3 = \frac{h}{-rh + 3\,h}$$
$$\alpha'_{\frac{7}{2}} = -\left(\frac{h}{-rh + 3\,h}\right)$$
$$\beta'_0 = \frac{1}{1440}\left(\frac{45\,h^6 - 35\,h^6 r - 34\,t - 2\,rh^6}{h^4}\right) \tag{2.5}$$
$$\beta'_1 = \frac{1}{360}\left(\frac{108\,h^6 + 48\,th^6}{h^4}\right)$$
$$\beta'_2 = \frac{1}{240}\left(\frac{189\,h^6 - 12\,h^6 r - 82\,th^6 + 16\,rh^6}{h^4}\right)$$
$$\beta'_3 = \frac{1}{360}\left(\frac{144\,h^6 + 376\,th^6 + 5\,h^6 r - 13\,rh^6}{h^4}\right)$$

$$\beta_4' = -\frac{1}{1440}\left(\frac{-12\,h^6r - 270\,th^6 + 2\,rh^6}{h^4}\right)$$

simplifying the result gives a discrete hybrid linear multistep method

$$\begin{aligned} y_{n+4} &= \left(\frac{-rh+th+3\,h}{-rh+3\,h}\right)y_{n+3} + \left(-\frac{th}{-rh+3\,h}\right)y_{n+r} + \\ &\frac{1}{1440}\left(\frac{45\,th^6 - 147\,th^6r + 112\,th^6r - 17\,t^2h^6 - 2\,trh^6}{h^4}\right)f_n + \\ &\frac{1}{360}\left(\frac{108\,th^6 - 89\,th^6r + 24\,t^2h^6 - 19\,trh^6}{h^4}\right)f_{n+1} + \\ &\frac{1}{240}\,\frac{189\,th^6 - 12\,th^6r - 41\,t^2h^6 - 41\,th^6r + 16\,trh^6}{h^4})f_{n+2} + \\ &\frac{1}{360}\left(\frac{144\,th^6 + 188\,t^2h^6 + 5\,th^6r - 13\,trh^6}{h^4}\right)f_{n+3} \\ &-\frac{1}{1440}\left(\frac{27\,th^6 - 12\,th^6r - 135\,t^2h^6 + 19\,th^6r - 12\,trh^6 + 2\,trh^6}{h^4}\right)f_{n+4} \end{aligned} \tag{2.6}$$

Evaluating (2.4) and (2.5) at $t = 1$ gives a discrete hybrid linear multistep method.

$$\begin{aligned} y_{n+4} = &\frac{1}{1440h^3\,(3\,h-rh)}(-1440\,rh\,y_{n+3}h^3 + 5760\,y_{n+3}h^4 - 1440\,y_{n+r}h^4 + \\ &\left(2\,rh^6 + 324\,h^6 - 81\,rh\,h^5 - 60\,rh^3h^3 + 55\,rh^4h^2 - 18\,rh^5\right) f_{n+4} \\ &+ \left(320\,rh^3h^3 + 3984\,h^6 - 1508\,rh\,h^5 - 8\,rh^6 - 280\,rh^4h^2 + 84\,rh^5\right) f_{n+3} \\ &+ \left(2664\,h^6 + 570\,rh^4h^2 + 12\,rh^6 - 1050\,rh\,h^5 - 720\,rh^3h^3 - 144\,rh^5\right) f_{n+2} + \\ &\left(108\,rh^5h - 1932\,rh\,h^5 - 8\,rh^6 + 960\,rh^3h^3 + 1584\,h^6 - 520\,rh^4h^2\right) f_{n+1} + \\ &\left(2\,rh^6 + 84\,h^6 - 500\,rh^3h^3 + 720\,rh^2h^4 - 469\,rh\,h^5 + 175\,rh^4h^2 - 30\,rh^5\right) f_n) \end{aligned} \tag{2.7}$$

$$\begin{aligned} y'_{n+4} = &\frac{1}{1440h^4(3h-rh)}[1440\,y_{n+3}h^4 - 1440\,y_{n+r}h^4 + \\ &\left(2\,rh^6 + 55\,rh^4h^2 - 18\,rh^5h + 1425\,h^6 - 60\,rh^3h^3 - 448\,rh\,h^5\right) f_{n+4} + \\ &\left(5604\,h^6 - 8\,rh^6 + 84\,rh^5h + 320\,rh^3h^3 - 280\,rh^4h^2 - 2048\,rh\,h^5\right) f_{n+3} + \\ &\left(570\,rh^4 - 768\,rh\,h^5 + 12\,rh^6 - 144\,rh^5h + 1818\,h^6 - 720\,rh^3h^3\right) f_{n+2} \\ &+ \left(960\,rh^3h^3 - 2048\,rh\,h^5 + 1932\,h^6 + 108\,rh^5h - 8\,rh^6 - 520\,rh^4h^2\right) f_{n+1} + \\ &\left(720\,rh^2h^4 - 30\,rh^5h - 500\,rh^3h^3 - 448\,rh\,h^5 + 21\,h^6 + 2\,rh^6 + 175\,rh^4h^2\right) f_n] \end{aligned} \tag{2.8}$$

Simplifying (2.7) and (2.8) at $r = \frac{7}{2}$, we have:

$$y_{n+4} = 2\,y_{n+\frac{7}{2}} - y_{n+3} + \frac{h^2}{422400}(35024f_{n+4} + 105160f_{n+3} + 10560f_{n+1} - 48290f_{n+2} - 3465f_n) \tag{2.9}$$

$$y'_{n+4} = 2\,y_{n+3} - 2\,y_{n+\frac{7}{2}} - \frac{h^2}{23040h}\left(-2912\,f_{n+1} + 7146\,f_{n+2} - 14376\,f_{n+3} - 7663\,f_{n+4} + 525\,f_n\right) \tag{2.10}$$

The predictors and its first derivative are developed using power series as the basis function to obtain

$$y_{n+4} = -y_{n+3} + 2\,y_{n+\frac{7}{2}} + \frac{h^2}{1344000}\left(3900f_n - 21980f_{n+1} + 55300f_{n+2} - 83300f_{n+3} + 382080f_{n+\frac{7}{2}}\right)$$

$$\begin{aligned} y'_{n+4} = &y_{n+\frac{7}{2}} - y_{n+3} + \frac{h^2}{201600h}\left(4985\,f_n - 28161\,f_{n+1} + 71575\,f_{n+2} - 142415\,f_{n+3}\right. \\ &\left.+ 245216\,f_{n+\frac{7}{2}}\right) \end{aligned}$$

3 ANALYSIS OF THE BASIC PROPERTIES OF THE METHOD

3.1 Order of the Method

Let the linear operator L associated with the method (2.9) be defined as

$$L[y(x);h] = \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h^2 \beta_j y''(x_{n+jh})$$

where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval $[a, b]$. Expanding $y(x_n + jh)$ and $y''(x_n + jh), j = 0, 1, ..., m$ in

Taylor series about x_n and collecting like terms in h and y gives;

$$L[y(x);h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y^{(2)}(x) + \cdots + c_p h^p y^{(p)}(x) \quad (3.1)$$

Definition 1. The difference operator L associated with the discrete implicit one step method (3.1) are said to be of order p if in (3.2) $c_0 = c_1 = c_2 = \cdots = c_{p+1} = 0, c_{p+2} \neq 0$ see [1]

Definition 2. The term c_{p+2} is called the error constant and it implies that the local truncation error is given as

$t_{n+k} = c_{p+2} h^{(p+2)}(x_n) = 0(h^{p+3})$ see [1]

Definition 3. Linear Multistep method (LMM) is a computational method for determining the sequence y_n which takes the form of a linear relationship between y_{n+j} and f_{n+j},$j = 0(1)k$. The general form of a linear k- step method for mth order general odes may be written as

$$y(x) = \sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j},$$

α_j,β_j are the coefficients of the method, f_{n+j}=$f(x_{n+j}$,y_{n+j},y^i_{n+j},y^{ii}_{n+j},$\cdots$,$y^{m-1}_{n+j})$, $j = 0(1)k$, h is the steplength, m is the order of ode to be solved: $\alpha_k \neq 0$. α_0 and β_0 are not both zero. [1]

Definition 4. A multistep method is said to be P-satble, if its interval of periodicity is $(0, \infty)$ see [13]

Order and Error constant of the Methods are as follows Applying the linear operator L (3.1) to determine the order and the error constants of the derived method. Expanding the method (2.9) and its derivative (2.10) by Taylor's series and combining the coefficient of the like terms in h^n gives

$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_5, c_7 = \frac{-89}{15360} = 5.794e^{-3}$

Hence, the method is of order 5 with error constant $c_7 = 5.794e^{-3}$

and

$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_7 = \frac{5293}{14} = 378.07$

Hence, the method is of order 5 with error constant $c_7 = 378.07$

Zero Stability of the Method

Given the first characteristics polynomial of (2.9) as:

$$\rho(r) = r^4 - 2r^{\frac{7}{2}} + r^3 = 0$$

On solving $\rho(r)$, $r = 0, 1$ which satisfies $|R_J| \geq 1, j = 1, \ldots, k$. That the roots lies in the unit circle and the multiplicity is simple. Hence the method is zero stable.

Consistency of the Method

A numerical method is said to be consistent if the following conditions are satisfies

(i) the order $p \geq 1$
(ii)$\sum_{j=0}^{k} \alpha_j = 0$
(iii) $\rho(1) = \rho'(1) = 0$
(iv)$\rho''(1) = 2!\sigma(1)$

where, $\rho(r)$ and $\sigma(r)$ are the first and second characteristics polynomials of our method. According to [1], the first condition is a sufficient condition for the associated block method to be consistent. Our method is order $p = 5$. Hence it is consistent.

Region of Absolute Stability of the Method

Consider the stability polynomial

$$\prod(z,\bar{h}) = \rho(z) - \bar{h}\sigma(z) = 0 \qquad (3.2)$$

To determine the region of absolute stability in this work, a method that requires neither the computation of roots of a polynomial nor the solving of simultaneous inequality was adopted. This methods according to [1] is called the Boundary Locus Method (BLM).

Definition 5. The region R of the complex $\bar{h}-$ plane such that the roots of $\prod(r,\bar{h}) = 0$ lie within the unit circle whenever $\bar{h}$ lies in the interior of the region is called the region of absolute stability.

Thus, we redefine (3.3) in terms of Euler's number, $e^i\theta$, as follows

$$\pi(e^{i\theta}, h) = \rho(e^{i\theta}) = 0 \qquad (3.3)$$

so that, the locus of the boundary δR is given by

$$\bar{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} \qquad (3.4)$$

where ρ is the first characteristics polynomial and σ is second characteristics polynomial

if $\prod(z,\bar{h}) = 0, \bar{h} = \lambda h^2$

then

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \qquad (3.5)$$

$$\rho(r) = r^4 - 2r^{\frac{7}{2}} + r^3$$

$$\sigma(r) = \frac{1}{422400}(35024r^4 + 105160r^3 - 48290r^2 + 10560r - 3465)$$

Using Matlab to plot (3.5) gives the required stability region below

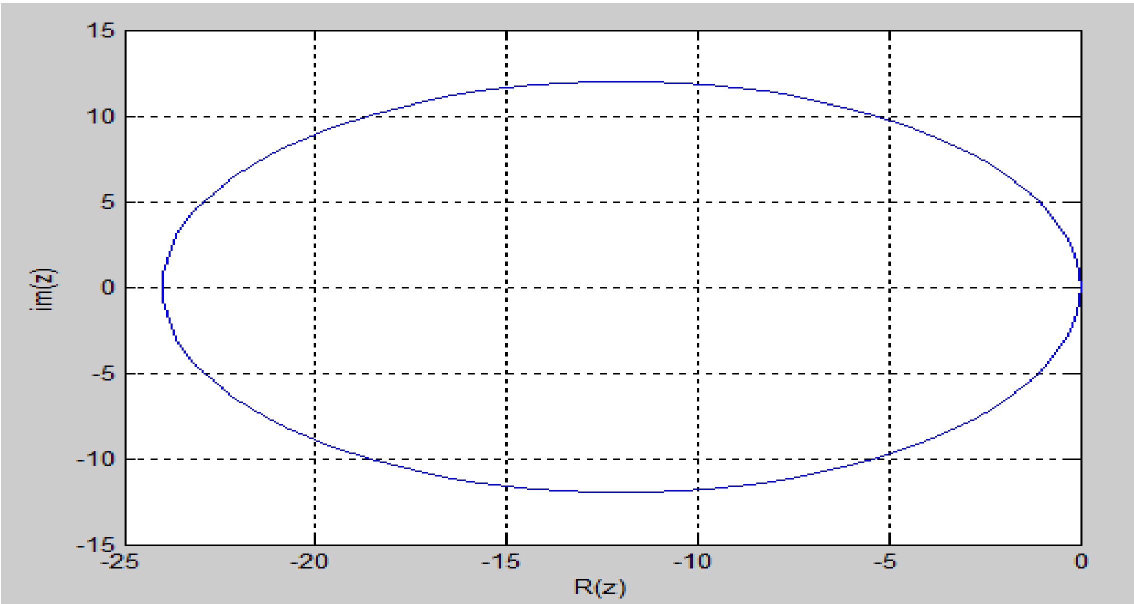

Fig. 1. Region of Absolute Stability of the method.

As shown on the graph, the region of absolute stability of the method is (-24,0). The method is P-stable .

4 NUMERICAL EXAMPLES

Test problems

Problem 1

$y'' = y'$: $y(0) = 0$; $y'(0) = -1$; $h = 0.1$

exact solution: $y(x) = 1 - exp(x)$

Problem 2.

$y'' = 100y$; $y(0) = 1, y'(0) = -10, h = 0.01$

exact solution:$y(x) = \exp(-10x)$

Problem 3:

Resonance Vibration of a Machine

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger which moves vertically up and down by a fly wheel makes the impact force on the metal sheet and therefore the supporting base, intermittent and cyclic. The bearing base on which the metal sheet is situated has a mass, $M = 2000kg$. The force acting on the base follows a function: $f(t) = 2000sin(10t)$,in which t=time in seconds. The base is supported by an elastic pad with an equivalent spring constant $k = 2 * 10^5 N/M$. Determine the differential equation for the instantaneous position of the base $y(t)$ if the base is initially depressed down by an amount 0.1m.

Solution: The mass- spring system above is modeled as differential equation:

The Bearing base mass = 2000kg

Spring constant $k = 2 * 10^5 N/m$

Force (ma) on the metal sheet= $m\frac{d^2y}{dt^2} = my''$

i.e. $ma = my'' = 2000sin(10t)$; where $a = y''$

Initial conditions on the system are

$y(t_0) = y_0; \frac{dy}{dt}|t = 0 = y'(t_0) = y'(0); t_0 = 0, y'_0 = 0.1$

Therefore, the governing equation for the instantaneous position of the base y(t) is given by

$My'' + ky = F(t); y(t_0) = y_0, y'(t_0) = y'0$

Theoretical solution: $y(t) = \frac{1}{10}cos10t + \frac{1}{200}sin10t - \frac{t}{20}cos10t$

4.1 Results

Table 1. Result of problem 1, for h=0.1

X	Exact solution	Computed solution	Error in new method	Error in [11]
0.2	-0.221402758160170	-0.221402754700000	$3.46017000e-09$	$8.17176E-07$
0.3	-0.349858807576003	-0.349858801900000	$5.67600300e-09$	$3.10356E-06$
0.4	-0.491824697641270	-0.491824690000000	$7.64127000e-09$	$6.56957E-06$
0.5	-0.648721270700128	-0.648721260202993	$1.04971347e-08$	$1.14380E-05$
0.6	-0.822118800390509	-0.822118785895474	$1.44950355e-08$	$1.79656E-05$
0.7	-1.013752707470477	-1.013752688688239	$1.87822380e-08$	$2.64474E-05$
0.8	-1.225540928492468	-1.225540905693596	$2.27988702e-08$	$3.72222E-05$
0.9	-1.459603111156950	-1.459603082898740	$2.82582100e-08$	$5.06788E-05$
1.0	-1.718281828459045	-1.718275773210793	$3.55473540e-08$	$6.72615E-05$

Table 2. Result of problem 3

X-value	y-exact	y- computed	Error in the new method	Error in [12]
0.10	0.3678794411714423	0.3678794504846674	$8.1951512e-09$	1.157e-7
0.20	0.3328710836980796	0.3328710939166178	$9.3132251e-09$	3.658e-7
0.30	0.3011942119122021	0.3011942232819327	$1.02185382e-08$	6.051e-7
0.40	0.2725317930340126	0.2725318050644373	$1.20304247e-08$	8.502e-7
0.50	0.2465969639416065	0.2465969772327199	$1.32911134e-08$	1.104e-6
0.60	0.2231301601484298	0.2231301746281558	$1.44797260e-08$	1.369e-6
0.70	0.2018965179946554	0.2018965339041756	$1.59095202e-08$	1.450e-6
0.80	0.1826835240527347	0.1826835411462089	$1.70934742e-08$	1.597e-6
0.90	0.1652988882215865	0.1652989069915710	$1.87699845e-08$	1.763e-6
1.00	0.1495686192226351	0.1495686397195046	$2.04968695e-08$	1.946e-6

Table 3. Computed Results and error of Problem 3

t	exact solution	Computed solution	Error
0.01	0.099404629653415691	0.099404631194783169	$1.541367e-09$
0.02	0.097958005773976925	0.097958008750775830	$2.976799e-09$
0.03	0.095207162458893865	0.095207163562179453	$1.032860e-09$
0.04	0.091970827382988077	0.091970829830720871	$2.447733e-09$
0.05	0.087961427477332363	0.087961431546444119	$4.069112e-09$
0.06	0.082363909854646533	0.082363923228371022	$1.337372e-08$
0.07	0.076833743309093400	0.075850410853356878	$1.711910e-08$
0.08	0.069604876901833215	0.069604890787124438	$1.388529e-08$
0.09	0.062811758617177721	0.062811776948016179	$1.833084e-08$
0.10	0.055536073981512724	0.055536116843483072	$4.286197e-08$

5 DISCUSSION

In this paper, we have considered three numerical examples to test the efficiency of our method. First problem was solved by [11] while the second problem was solved by [12], the third problem is Engineering problem. The new method gave better approximation because the proposed method require starting value and does not require self-starting.

6 CONCLUSION

In this paper, we have proposed a Predictor-Corrector method for the solution of second order ordinary differential equations. Our method was found to be zero stable, consistent and converges. The numerical examples show that our method gave better accuracy than the existing methods.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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