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On Convexity of Right-Closed Integral Sets

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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Short Research Article

Abstract

Let $\mathcal{N}$ denote the set of non-negative integers. A set of non-negative, n-dimensional integral vectors, $\mathcal{M} \subset \mathcal{N}^n$, is said to be *right-closed*, if $((\mathbf{x} \in \mathcal{M}) \wedge (\mathbf{y} \geq \mathbf{x}) \wedge (\mathbf{y} \in \mathcal{N}^n)) \Rightarrow (\mathbf{y} \in \mathcal{M})$.
In this paper, we present a polynomial time algorithm for testing the convexity of a right-closed set of integral vectors, when the dimension n is fixed. Right-closed set of integral vectors are infinitely large, by definition. We compute the convex-hull of an appropriately-defined finite subset of this infinite-set of vectors. We then check if a stylized *Linear Program* has a non-zero optimal value for a special collection of facets of this convex-hull.
This result is to be viewed against the backdrop of the fact that checking the convexity of a real-valued, geometric set can only be accomplished in an approximate sense; and, the fact that most algorithms involving sets of real-valued vectors do not apply directly to their integral counterparts. This observation plays an important role in the efficient synthesis of Supervisory Policies that avoid Livelocks in *Discrete-Event/Discrete-State Systems.*

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1 Introduction and Motivation

A set in Euclidean space is called convex if for every pair of points in the set, the line segment connecting them, lies completely in the set. While a well-established concept, unfortunately checking this property for an arbitrary geometric set is not easy. For example, consider deleting a single point from a convex set; determining that the set is non-convex is almost impossible as infinite pairs of points should be tested [1].

The concept of convexity lends itself to several applications in engineering [2], and it also appears in the context of supervisory control of *Discrete Event/Discrete State* (DEDS) systems. Manufacturing Systems, multicomponent systems with event-driven dynamics such as shipyards, airports, claims department of insurance offices are examples of DEDS systems. The discrete-states of these systems have a logical, as opposed to numerical, interpretation. A DEDS system is *live* [3], if irrespective of the past activities, every event can be executed, not necessarily immediately, in the future. A *live-locked* DEDS system will have at least one event that enters a state of suspended animation for perpetuity, which is undesirable for various reasons. A live DEDS system does not experience live-locks.

Petri nets (PNs) are a popular modeling paradigm for DEDS systems. A PN model of a DEDS system that is not live originally, can be made live by a *supervisory policy*. This supervisory policy is characterized by a right-closed set of integral vectors (cf. references [4], [5], [6], [7], and [8] for additional information), and when this set is convex, the supervisory policy that enforces liveness can be implemented inexpensively (cf. references [9], [10]). This is the motivation behind testing the convexity of an arbitrary right-closed set, which is introduced in the remainder of this section.

The notion of convexity in real sets [11] can be extended to the integer sets as well. An integral set is *integer-convex* if for every pair of integer points, all the lattice points on the line segment connecting them also belong to the set. The other definition could be that an integer set is convex if it is equal to the set of all the lattice points in a real, convex set. These two definitions are not equivalent in the context of integral sets. This difference makes most of the convexity checking algorithms ineffective over integral sets. Comparing the number of lattice points in the convex hull of the integral set with the number of lattice points in the original integral set could be a good starting point for any convexity testing algorithm. But the complexity of lattice point enumeration will increase exponentially with the number of integer points inside the hull [12], [13]. Constructing the convex hull of k points in a fixed dimension can be done in polynomial time [14]. As discussed earlier right-closed set is unbounded, therefore constructing the convex hull of a right-closed integral set will be computationally infeasible. Instead, we use a finite subset of the original right-closed set that keeps the characteristics we need to test the convexity of the entire set. Using the convex hull of this finite subset, we will derive a polynomial time convexity test for an arbitrary right-closed set of integral vectors.

The remainder of the paper is organized as follows. Preliminary definitions and notations will be introduced in the next section. The main theorems and results will be presented on Section 3. The details of the algorithm are provided in section 4. Section 5 concludes the paper.

2 Notations and definitions

Throughout this paper $\mathcal{N}(\mathcal{N}^+)$ and $\mathcal{R}$ will be used to denote the set of non-negative (positive) integers and reals, respectively. For $m, n, k \in \mathcal{N}^+$, the set of $m \times n$ matrices (k-dimensional vectors) of non-negative integers is represented as $\mathcal{N}^{m\times n}$ ($\mathcal{N}^k$). Similarly, $\mathcal{R}^{m\times n}$ and $\mathcal{R}^k$ denotes the set of $(m \times n)$ real-valued matrices and k-dimensional real-valued vectors, respectively.

The *convex combination* of a set of integral- or real-valued vectors, $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$ is defined as $\sum_k^{i=1} \lambda_i \mathbf{x}_i$, where (1) $0 \le \lambda_1, \lambda_2, \cdots, \lambda_k \le 1$, and (2) $\sum_k^{i=1} \lambda_i = 1$. If the latter condition is dropped, then the combination is called *conic combination* and if only the later property holds, it is an *affine combination*. The *convex hull* of these vectors is defined as the set of all possible convex combinations of the vectors which is the smallest, real-valued, convex set that contains these vectors, and it is denoted by $conv(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k)$.

A set $\mathcal{M} \subseteq \mathcal{R}^n$ is called convex if for every pair of points in $\mathcal{M}$, the line segment connecting the pair also lies in $\mathcal{M}$. This definition can be extended into two distinct statements for integral sets:

1. The integer set $\mathcal{M} \in \mathcal{N}^n$ is *segmentally* convex if all the integral points on the line segment connecting each pair of points in $\mathcal{M}$ also belongs to $\mathcal{M}$.
2. The integer set $\mathcal{M} \in \mathcal{N}^n$ is *intersection* convex if there is a real convex set C such that its set of integer points equals to $\mathcal{M}$.

Although equivalent for real sets, these two notions of convexity are not necessarily similar for an integral set. The *intersection* convexity of an integral set implies the *segmental* convexity of that set, but the converse is not true. Hence, throughout this paper, we use the term convexity to refer to the intersection convexity of a set.

An integral set of vectors $\mathcal{M} \subseteq \mathcal{N}^n$ is said to be *right-closed* or *upward-closed* [15] if $((\mathbf{m}^1 \in \mathcal{M}) \wedge (\mathbf{m}^2 \ge \mathbf{m}^1) \wedge (\mathbf{m}^2 \in \mathcal{N}^n))$, would imply $\mathbf{m}^2 \in \mathcal{M}$. Every right-closed set $\mathcal{M}$ contains a finite set of minimal elements, $\min(\mathcal{M}) \subset \mathcal{M}$ where

1. $\forall \mathbf{m}^1 \in \mathcal{M}$, there exists $\mathbf{m}^2 \in min(\mathcal{M})$ such that $\mathbf{m}^2 \le \mathbf{m}^1$, and
2. if $\exists \mathbf{m}^1 \in \mathcal{M}$, $\exists \mathbf{m}^2 \in min(\mathcal{M})$ such that $\mathbf{m}^2 \ge \mathbf{m}^1$ then $\mathbf{m}^1 = \mathbf{m}^2$.

A right-closed, real-valued set of vectors, that is closed, is defined analogously. However, unlike an integral right-closed set, the set of minimal elements of a real-valued right-closed set is not finite.

For $i, j \in \mathcal{N}$, $\mathbf{A} \in \mathcal{R}^{i\times j}$, $\mathbf{b} \in \mathcal{R}^i$, a *polyhedron* $P(\mathbf{A}, \mathbf{b})$ is described as $P(\mathbf{A}, \mathbf{b}) := \{\mathbf{x} \in \mathcal{R}^n | \mathbf{A}\mathbf{x} \ge \mathbf{b}\}$. If the entries of $\mathbf{A}$ and $\mathbf{b}$ are rational numbers, then $P(\mathbf{A}, \mathbf{b})$ is a *rational polyhedron*. The set of integral points inside the $P(\mathbf{A}, \mathbf{b})$ is denoted by $Int(P(\mathbf{A}, \mathbf{b}))$. A *polytope* or a bounded polyhedron is a convex hull of a finite set of vectors in $\mathcal{R}^n$.
A *half-space* $(\mathbf{w}, t)$ is the set $\{\mathbf{x} \in \mathcal{R}^n \mid \mathbf{w}^T\mathbf{x} \ge t\}$ for $\mathbf{w} \in \mathcal{R}^n$, $t \in R$. We use the notation $\{(\mathbf{w}_i, t_i)\}_{i=1}^k$ to denote the intersection of a set of k-many half-spaces.

A half-space $(\mathbf{w}, t)$ is a *valid inequality* for a set $\mathcal{S} \subset \mathcal{R}^n$, if $\mathcal{S} \subset \{\mathbf{x} \in \mathcal{R}^n \mid \mathbf{w}^T\mathbf{x} \ge t\}$. $\mathcal{F}$ is a *face* of the polyhedron $P(A, b)$, if $\mathcal{F} \subset P(\mathbf{A}, \mathbf{b})$ and there exists a valid inequality $(\mathbf{w}, t)$ for $P(\mathbf{A}, \mathbf{b})$ such that $\mathcal{F} = \{\mathbf{x} \in P(\mathbf{A}, \mathbf{b}) \mid \mathbf{w}^T\mathbf{x} = t\}$. If $\mathcal{F} \ne \varnothing$, then $(\mathbf{w}, t)$ *supports* the face $\mathcal{F}$, and $\mathcal{F} = \{\mathbf{x} \in P(\mathbf{A}, \mathbf{b}) \mid \mathbf{w}^T\mathbf{x} = t\}$ is called the *supporting hyperplane* of $\mathcal{F}$. A supporting hyperplane is called *right-closed supporting hyperplane* if both $\mathbf{w}$ and t are non-negative. $\mathcal{F}$ is called *proper-face* if $\mathcal{F} \ne P(\mathbf{A}, \mathbf{b})$ and *non-trivial* if $\mathcal{F} \ne \varnothing$. A *facet* of the polyhedron $P(\mathbf{A}, \mathbf{b})$, $\mathcal{F}$, is the proper face of $P(\mathbf{A}, \mathbf{b})$ such that it is not strictly contained in any proper or non-trivial face of $P(\mathbf{A}, \mathbf{b})$. A *right-closed facet* of a polyhedron $P(\mathbf{A}, \mathbf{b})$ is a subset of a right-closed supporting hyperplane.

The procedure to compute the convex hull of a set of vectors is foundational to this paper. The following result shows that this can be done in polynomial time.

Theorem 2.1. *[14] It is possible to compute the convex hull of m points in n-space deterministically in $\mathcal{O}(mlog(m) + m^{\lfloor n/2 \rfloor})$*

For a right-closed $\mathcal{M} \subseteq \mathcal{N}^n$, $P(\mathbf{A}, \mathbf{b}) = conv(\mathcal{M})$, if and only if $\mathbf{A}$ and $\mathbf{b}$ are non-negative [9]. The following lemma identifies a property of the convex hull of $\min(\mathcal{M})$ where $\mathcal{M} \subseteq \mathcal{N}^n$ is an arbitrary right-closed set.

Lemma 2.2. *[9] If $\mathcal{M} \subseteq \mathcal{N}^n$ is an arbitrary right-closed set, then all vertices of the convex hull of $\min(\mathcal{M})$ are minimal elements.*

This observation follows directly from the fact that every vector in $conv(\min(\mathcal{M}))$ is a convex combination of the finite set of vectors in $min(\mathcal{M})$. We turn our attention to the problem of checking integer convexity of a right-closed set in the next section.

3 Main Results

As theorem 2.1 states, computing the convex hull of finite points in a finite dimension has a polynomial-time complexity with respect to number of points for a fixed dimension. We will utilize this theorem as a basis for our proposed algorithm presented in the next section. As the right-closed set of integral vectors is an infinite set by definition, constructing its convex hull will be computationally infeasible. To overcome this issue, we propose a finite subset of the original right-closed set that keeps the characteristics that can attest to the convexity of the original right-closed set.

Let $\mathcal{V} \subset \mathcal{M}$ be a finite collection of vectors as defined below:

$$\mathcal{V} = min(\mathcal{M}) \cup \left\{ \bigcup_{p \in \{1,\cdots,k\}, q \in \{1,\cdots,n\}} \{\mathbf{m}_p + \mathbf{1}_q\} \right\} \tag{3.1}$$

where $\min(\mathcal{M}) = \{\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_k\}$ and $\{\mathbf{1}_q\}_{q=1}^n$ is the set of n-many unit-vectors. In general, $\min(\mathcal{M}) \subseteq \min(Int(conv(\mathcal{V})))$. From Lemma III.4 of reference [9], we note that for a non-convex right-closed set $\mathcal{M} \subseteq \mathcal{N}^n$, $\min(Int(conv(\mathcal{V}))) - \mathcal{M} \neq \emptyset$.

The finite set $\mathcal{V}$ is a strict subset of $\mathcal{M}$, hence we can infer that the set of supporting hyperplanes defining $conv(\mathcal{M})$ is a strict subset of supporting hyperplanes defining $conv(\mathcal{V})$. On the other hand, $conv(\mathcal{M})$ is a right-closed set which means all its supporting hyperplanes are right-closed hyperplanes. The following theorem shows that if a supporting hyperplane of $conv(\mathcal{V})$ is not a supporting hyperplane of $conv(\mathcal{M})$, then it is not right-closed.

Theorem 3.1. *The set of right-closed supporting hyperplanes defining $conv(\mathcal{V})$ are the only supporting hyperplanes defining $conv(\mathcal{M})$.*

Proof. Since $\mathcal{M} \subseteq \mathcal{N}^n$ is right-closed, $conv(\mathcal{M}) \subseteq \mathcal{R}^n$ is also right-closed. From reference [9], $\exists l \in \mathcal{N}, \exists \mathbf{A} \in \mathcal{R}^{l \times n}, \exists \mathbf{b} \in \mathcal{R}^l$, such that $conv(\mathcal{M}) = P(\mathbf{A}, \mathbf{b})$. As $conv(\mathcal{V}) \subset conv(\mathcal{M})$, extra constraints should be added to the mentioned polyhedron in order to construct $conv(\mathcal{V})$. Let $\mathbf{c}_i^T\mathbf{x} \geq d_i$ for $i = 1, \cdots, K$, be the K additional constraints. For any vector $\mathbf{p} \in conv(\mathcal{M}) - conv(\mathcal{V})$, we obtain

$$\mathbf{c}_i^T\mathbf{p} < d_i, \forall i = 1, \cdots, K$$

Additionally, for any $\mathbf{y} \geq 0$, $\mathbf{p} + \mathbf{y}$ also does not belong to $conv(\mathcal{V})$ either. Therefore,

$$\mathbf{c}_i^T\mathbf{p} + \mathbf{c}_i^T\mathbf{y} < d_i, \forall i = 1, \cdots, K.$$

Comparing these two expressions, suggests that $\mathbf{c}_i$ for $i = 1, \cdots, K$ has to have at least one negative component, regardless of the sign of d_i. Hence, the additional constraints in the form of $\mathbf{c}_i^T\mathbf{x} \geq d_i$ cannot be right-closed hyperplanes □

For $1 \leq i \leq n$, let γ_i denote the maximum value of the i-th component of each member of $\mathcal{V}$. That is, $\gamma_i := \max\{\mathbf{x}_i \mid \mathbf{x} \in \mathcal{V})\}$. We define $\widetilde{P} \subseteq \mathcal{R}^n$ as the polytope that is defined as:

$$conv(\mathcal{M}) \cap \{(-\mathbf{1}_i, -\gamma_i)\}_{i=1}^{n} \tag{3.2}$$

That is, each of the left-closed half-spaces ensures the i-th component is less than or equal to γ_i. Therefore $\widetilde{P}$ consists of either right-closed or left-closed hyperplanes.

Theorem 3.2. *The set of right-closed supporting hyperplanes defining $conv(\mathcal{V})$ are the only right-closed supporting hyperplanes defining $conv(\mathcal{M})$ and $\widetilde{P}$*

Proof. We first show that $conv(\mathcal{V}) \subset \widetilde{P}$. Notice that $conv(\mathcal{V}) \subset conv(\mathcal{M})$. Also all the points in $conv(\mathcal{V})$ satisfy the previously mentioned left-closed hyperplanes that are added to $conv(\mathcal{V})$ in order to construct the $\widetilde{P}$. Hence, every point in $conv(\mathcal{V})$ also belongs to $\widetilde{P}$.

Now we show that $conv(\mathcal{V})$ is constructed from $\widetilde{P}$ by adding only non-right closed hyperplanes. Let $cx \geq d$ be the constraint added to $\widetilde{P}$ in order to construct the $conv(\mathcal{V})$. For every $v \in conv(\mathcal{V})$ we have $cv \geq d$. Define $\gamma_{max} = (\gamma_1, \cdots, \gamma_n)$. If c and d are to be non-negative, therefore $c\gamma_{max} \geq d$ as $\gamma_{max} \geq v$. But $\gamma_{max} \notin conv(\mathcal{V})$ by its definition although $\gamma_{max} \in \widetilde{P}$. Therefore $c\gamma_{max} < d$. This contradicts the assumption that both c, d are non-negative. Hence, the additional constraints to $\widetilde{P}$ cannot be right-closed. □

Therefore, the (half-space) description of $\widetilde{P}$ can be obtained by adding additional half-spaces $\{(-1, \gamma_i)\}_{i=1}^{n}$ to the set of right-closed facets of $\mathcal{V}$. This presents an computational procedure for constructing the polytope $\widetilde{P}$. It is straightforward to show that $\min(Int(conv(\mathcal{M}))) = \min(Int(\widetilde{P}))$.

Theorem 3.3. *$\forall \boldsymbol{x} \in min(Int(\widetilde{P}))$, there is at least one right-closed facet of $\widetilde{P}$, $\mathcal{F}$, such that $\alpha\boldsymbol{x} \in \mathcal{F}$ for some $0 \leq \alpha \leq 1$.*

Proof. The $\widetilde{P}$ is bounded (by definition) and therefore a compact space. Let $P(\mathbf{A}, \mathbf{b})$ be the polyhedral representation of $\widetilde{P}$, which is the intersection of right- and left-closed half-spaces. Therefore, for $\forall \mathbf{x} \in \widetilde{P}$,

$$\alpha^* = \underset{0 \leq \alpha \leq 1}{min} \{\alpha \mid \alpha\mathbf{x} \ \in \widetilde{P}\}$$

is defined uniquely. Let right-closed rows of the $P(\mathbf{A}, \mathbf{b})$ (non-negative rows) be represented by $(\widehat{\mathbf{A}}, \widehat{\mathbf{b}})$, then for any $\mathbf{x} \in \widetilde{P}$, α^* can be defined uniquely as following:

$$\alpha^* = max \left\{ \frac{\widehat{\mathbf{b}}_i}{\widehat{\mathbf{A}}_{i,\cdot}\mathbf{x}} \right\}$$

where $\widehat{\mathbf{A}}_{i,\cdot}$ and $\widehat{\mathbf{b}}_i$ are the $i-$th row of the matrix $\widehat{\mathbf{A}}$ and vector $\widehat{\mathbf{b}}$ respectively.

This selection of α^* will guarantee that $(\alpha^*\mathbf{x})$ will satisfy at least one of the right-closed hyperplanes equation and also be in the feasible region defined by other right- and left-closed hyperplanes. Hence, for the aforementioned α^*, the $(\alpha^*\mathbf{x})$ satisfies all the constraints (rows) of $P(\mathbf{A}, \mathbf{b})$, so $(\alpha^*\mathbf{x}) \in P(\mathbf{A}, \mathbf{b}) = \widetilde{P}$, which suggests that there exist at least one right-closed facet of $\widetilde{P}$, $\mathcal{F}$, such that $\alpha^*\mathbf{x} \in \mathcal{F}$. □

Now we are ready to prove the main result of the paper.

Theorem 3.4. *A right-closed set $\mathcal{M}$ is convex if and only if $\forall \boldsymbol{x} \in \mathcal{F}_i$, where $\mathcal{F}_i$ is a member of set of right-closed facets defining the $\widetilde{P}$, $\lceil \boldsymbol{x} \rceil \in \mathcal{M}$.*

Proof. *(Only if)* Suppose $\exists \mathbf{x} \in \mathcal{F}_i$ ($\Rightarrow \mathbf{x} \in \widetilde{P}$) and $\lceil \mathbf{x} \rceil \notin \mathcal{M}$. There are two different scenarios:

1. either $\lceil \mathbf{x} \rceil$ is a member of $\widetilde{P}$ and as $\lceil \mathbf{x} \rceil \notin \mathcal{M}$, $\mathcal{M}$ is not convex, or
2. there is another minimal element of $\widetilde{P}$, $\tilde{\mathbf{m}}$, which is smaller than $\lceil \mathbf{x} \rceil$. $\tilde{\mathbf{m}}$ cannot belong to $\mathcal{M}$ either, because its membership to $\mathcal{M}$ implies $\lceil x \rceil \in \mathcal{M}$; Therefore $\mathcal{M}$ is not convex.

(If) Suppose $\mathcal{M}$ is not convex, then from the fact that $\min(Int(conv(\mathcal{M}))) = \min(Int(\widetilde{P}))$, and Lemma III.4 of reference [9], $\exists \mathbf{x} \in \min(\widetilde{P})$ such that $\mathbf{x} \notin \mathcal{M}$. From theorem 3.3, $\exists \alpha^*$ such that $0 \leq \alpha^* \leq 1$, $\exists$ a facet $\mathcal{F}_i$ that is on a right-closed hyperplane of $\widetilde{P}$, such that $\alpha^* \mathbf{x} \in \mathcal{F}_i$. Since $\alpha^* \mathbf{x} \leq \mathbf{x}$, $\mathbf{x} \notin \mathcal{M}$, and $\mathcal{M}$ is right-closed, it follows that $\alpha^* \mathbf{x} \notin \mathcal{M}$ as well; since $\lceil \alpha^* \mathbf{x} \rceil \leq \alpha^* \mathbf{x}$, $\lceil \alpha^* \mathbf{x} \rceil \notin \mathcal{M}$ □

Essentially theorem 3.4 notes that

$$(\forall \text{ right-closed facets } \mathcal{F}_i \text{ of } \widetilde{P}, \forall \mathbf{x} \in \mathcal{F}_i, \lceil \mathbf{x} \rceil \in \mathcal{M}) \Leftrightarrow (\mathcal{M} \text{ is convex}). \tag{3.3}$$

This leads to a polynomial time algorithm for verifying the convexity of a right-closed set of integral vectors, which is presented in the next section.

4 Algorithm

Each right-closed facet $\mathcal{F}_i$ is essentially defined by a collection of vertices. Each facet-defining vertex is either some $\mathbf{m}_i \in \min(\mathcal{M})$; or some $\widetilde{\mathbf{m}}_i \geq \mathbf{m}_i$, for some $\mathbf{m}_i \in \min(\mathcal{M})$. Let

$$\Upsilon(\mathcal{F}_i) := \{\mathbf{m}_i \in \min(\mathcal{M}) \mid \text{Either } \mathbf{m}_i, \text{ or some } \widetilde{\mathbf{m}}_i \geq \mathbf{m}_i \text{ is a vertex that defines } \mathcal{F}_i\}. \tag{4.1}$$

The following result uses the set $\Upsilon(\mathcal{F}_i)$ in the test identified in theorem 3.4.

Theorem 4.1.

$$(\forall \textit{ right-closed facets } \widetilde{\mathcal{F}}_i \textit{ of } \widetilde{P}, \forall \widetilde{\mathbf{x}} \in conv(\Upsilon(\widetilde{\mathcal{F}}_i)), \lceil \widetilde{\mathbf{x}} \rceil \in \mathcal{M}) \Leftrightarrow$$
$$(\forall \textit{ right-closed facets } \mathcal{F}_i \textit{ of } \widetilde{P}, \forall \mathbf{x} \in \mathcal{F}_i, \lceil \mathbf{x} \rceil \in \mathcal{M}).$$

Proof. ($\Rightarrow$ Part) Suppose $\exists$ a right-closed facet $\mathcal{F}_i$ of $\widetilde{P}$, and $\exists \mathbf{x} \in \mathcal{F}_i$, such that $\lceil \mathbf{x} \rceil \notin \mathcal{M}$. $\mathbf{x} \in \mathcal{F}_i$ can be written as the convex combination of the vertices for the given facet. By definition of the $\mathcal{F}_i$ in theorem 3.2, we can infer that a vertex of $\mathcal{F}_i$ is either a minimal elements of original right-closed set, or a vector $\mathbf{v}_i$ of the form: $\mathbf{v}_j = \mathbf{m}_j + \mathbf{y}_j$, for some $\mathbf{y}_j \in \mathcal{N}^n$ such that $\mathbf{y}_j \geq \mathbf{0}$. Hence,

$$\lceil \mathbf{x} \rceil = \left\lceil \sum_i \lambda_i \mathbf{m}_i + \sum_j \mu_j \mathbf{v}_j \right\rceil = \left\lceil \sum_i \lambda_i \mathbf{m}_i + \sum_j \mu_j (\mathbf{m}_j + \mathbf{y}_j) \right\rceil,$$

where $\sum_i \lambda_i + \sum_j \mu_j = 1$. If $\widetilde{\mathbf{x}} = \sum_i \lambda_i \mathbf{m}_i + \sum_j \mu_j \mathbf{m}_j$, then $\widetilde{\mathbf{x}} \in conv(\Upsilon(\mathcal{F}_i))$, $\widetilde{\mathbf{x}} \leq \mathbf{x} (\Rightarrow \lceil \widetilde{\mathbf{x}} \rceil \leq \lceil \mathbf{x} \rceil)$, and since $\mathcal{M}$ is right-closed, $\lceil \widetilde{\mathbf{x}} \rceil \notin \mathcal{M}$.

($\Leftarrow$ Part) Suppose $\exists$ a right-closed facet $\widetilde{\mathcal{F}}_i$ of $\widetilde{P}$, and $\exists \widetilde{\mathbf{x}} \in \widetilde{\mathcal{F}}_i$, such that $\lceil \widetilde{\mathbf{x}} \rceil \notin \mathcal{M}$. Since $\mathcal{M}$ is right closed, the set of integral vectors in $conv(\mathcal{M})$ is also right closed (cf. Lemma III.5, [9]). Additionally, $\widetilde{\mathbf{x}} \in conv(\mathcal{M})$, $\widetilde{\mathbf{x}} \leq \lceil \widetilde{\mathbf{x}} \rceil (\Rightarrow \lceil \widetilde{\mathbf{x}} \rceil \in conv(\mathcal{M}))$, and since $\lceil \widetilde{\mathbf{x}} \rceil \notin \mathcal{M}$, it follows that $\mathcal{M}$ is not convex. From theorem 3.4, $\exists$ a right-closed facet $\mathcal{F}_i$ of $\widetilde{P}$, and $\exists \mathbf{x} \in \mathcal{F}_i$, such that $\lceil \mathbf{x} \rceil \notin \mathcal{M}$. □

As a consequence of theorem 4.1, equation 3.3 can be written equivalently as

$$(\forall \text{ right-closed facets } \widetilde{\mathcal{F}}_i \text{ of } \widetilde{P}, \forall \widetilde{\mathbf{x}} \in conv(\Upsilon(\widetilde{\mathcal{F}}_i)), \lceil \widetilde{\mathbf{x}} \rceil \in \mathcal{M}) \Leftrightarrow (\mathcal{M} \text{ is convex}). \tag{4.2}$$

This observation leads to the following corollary.

Corollary 4.2. *If $\mathcal{M} \subseteq \mathcal{N}^n$ is a right-closed set such that $\min(\mathcal{M}) \subseteq \{0,1\}^n$, then $\mathcal{M}$ is convex.*

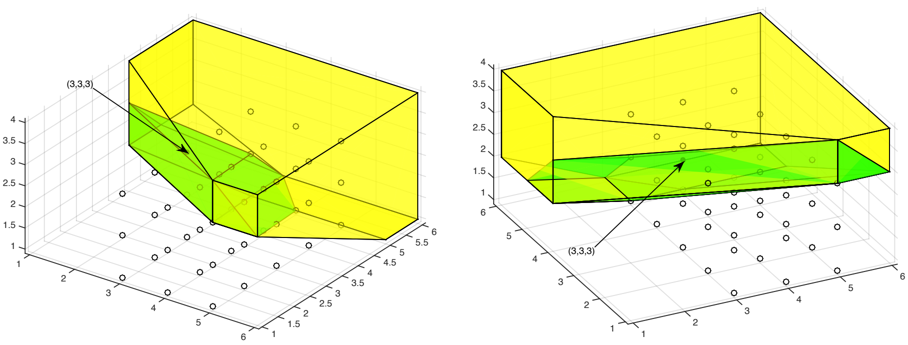
We present an efficient procedure that checks the condition $(\forall \widetilde{\mathbf{x}} \in conv(\Upsilon(\mathcal{F}_i)), \lceil \widetilde{\mathbf{x}} \rceil \in \mathcal{M})$ in the next subsection.

4.1 On testing the condition of Theorem 4.1

The above mentioned test condition requires us to find the ceiling function for the convex combination of the members of $\Upsilon(\mathcal{F}_i)$. Instead of going through all the points inside the convex hull of $\Upsilon(\mathcal{F}_i)$, one approach can be to create a "box" of integral points that are proximal to each right-closed facet. This is followed by verifying if each of these proximal vectors can be represented as a ceiling of convex combination of members of $\Upsilon(\mathcal{F}_i)$, using the feasibility LP described in theorem 4.3. Fig. 1 illustrates this approach for a right-closed set $\mathcal{M} \subseteq \mathcal{N}^3$ where

$$\min(\mathcal{M}) = \{(1\ 4\ 2)^T, (3\ 5\ 1)^T, (5\ 1\ 3)^T\}.$$

The polytope $\widetilde{P}$ has a right-closed facet $\mathcal{F}_1$ identified by the supporting hyperplane $x+3y+5z = 23$, where $\Upsilon(\mathcal{F}_1) = \{(5\ 1\ 3)^T, (3\ 5\ 1)^T, (1\ 4\ 2)^T\}$. The number of integral points in this box are determined by the components of vectors that constitute the $\Upsilon(\mathcal{F}_1)$-set. When the dimension is fixed, this number is polynomially related to its defining elements. As a result, the feasibility LP is executed a polynomial number of times, which in turn results in a polynomial time algorithm for testing the convexity of a right-closed set. This informal description is formalized in the remainder of this section.



(a) Creating the "box" (View 1) (b) Creating the "box" (View 2)

Fig. 1. Illustration of LP-based algorithm 1 on a right-closed set $\mathcal{M} \subseteq \mathcal{N}^3$ where $\min(\mathcal{M}) = \{(1\ 4\ 2)^T, (3\ 5\ 1)^T, (5\ 1\ 3)^T\}$. This figure shows the "box" of integral points that are proximal to the right-closed facet $\mathcal{F}_1$ defined by the hyperplane $x + 3y + 5z = 23$. Vector $(3\ 3\ 3)^T = \lceil(3\ 2.5\ 2.5)^T\rceil$, shown above, is the only integral vector that is proximal to the right-closed facet identified above. Since $(3\ 3\ 3)^T \notin \mathcal{M}$, we conclude that $\mathcal{M}$ is not convex.

Under the test condition, we are required to find the ceiling function for the convex combination of the members of $\Upsilon(\mathcal{F}_i)$. The following theorem establishes the polynomial time solvability of a key membership question that is used to test the condition of theorem 4.1.

Theorem 4.3. *Let $\Upsilon(\mathcal{F}_i) = \{\boldsymbol{m}_1.\boldsymbol{m}_2, \ldots, \boldsymbol{m}_k\}$ for a right-closed facet $\mathcal{F}_i$ of $\widetilde{P}$, and $\widetilde{\mathbf{m}} \in \mathcal{N}^n$ be an integral vector. Then, $\widetilde{\mathbf{m}} = \lceil\sum_i^k \lambda_i \boldsymbol{m}_i\rceil$ for a set $\{\lambda_i\}_{i=1}^k$ such that $\forall i \in \{1, \ldots, k\}, 0 \leq \lambda_i \leq 1$ and*

$\sum_{i=1}^{k} \lambda_i = 1$, *if and only if the Linear Program (LP) returns an optimal value* $\alpha^* \neq 0$.

$$LP(\widetilde{\boldsymbol{m}}) : \max(\alpha)$$
$$\textit{subject to}$$
$$\begin{pmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \dots & \boldsymbol{m}_k \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} \geq (\widetilde{\boldsymbol{m}} - \boldsymbol{1}) + (\alpha \times \boldsymbol{1})$$
$$\begin{pmatrix} \boldsymbol{m}_1 & \boldsymbol{m}_2 & \dots & \boldsymbol{m}_k \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} \leq \widetilde{\boldsymbol{m}}$$
$$\sum_{i=1}^{k} \lambda_i = 1, \forall i \in \{1, \dots, k\}, 0 \leq \lambda_i \leq 1, \textit{ and } 0 \leq \alpha,$$

where $\mathbf{1} \in \mathcal{N}^n$ *is the vector of all ones.*

Proof. (*Only If*) Suppose $\widetilde{\mathbf{m}} = \lceil \sum_i^k \lambda_i \mathbf{m}_i \rceil$ for a set $\{\lambda_i\}_{i=1}^k$ such that $\forall i \in \{1, \dots, k\}, 0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^k \lambda_i = 1$. Then it follows that $\widetilde{\mathbf{m}} - \mathbf{1} < \sum_i^k \lambda_i \mathbf{m}_i \leq \widetilde{\mathbf{m}}$. It follows that

$$\alpha^* \geq \min_j \left\{ \left(\sum_{i=1}^{k} \lambda_i \mathbf{m}_i \right)_j - (\widetilde{\mathbf{m}}_j - 1) > 0, \right\}$$

where $(\bullet)_j$ denotes the j-th component of a vector argument.

(*If*) From the constraints of LP, it follows that $\alpha^* \leq 1$. If $\alpha^* > 0$ with a corresponding set – $\{\lambda_i\}_{i=1}^k$ – that is feasible for LP, we can infer that

$$\widetilde{\mathbf{m}} - \mathbf{1} < \sum_{i=1}^{k} \lambda_i \mathbf{m}_i \leq \widetilde{\mathbf{m}} \Rightarrow \left\lceil \sum_{i=1}^{k} \lambda_i \mathbf{m}_i \right\rceil = \widetilde{\mathbf{m}}.$$

□

For each facet $\mathcal{F}_i$ of $\widetilde{P}$, let us define

$$\mathcal{L}(\mathcal{F}_i) := \{\mathbf{x} \in \mathcal{N}^n \mid \exists \mathbf{y} \in \mathcal{F}_i \text{ where } \mathbf{x} = \lceil \mathbf{y} \rceil\}.$$

From theorem 4.1 the right-closed set $\mathcal{M} \subseteq \mathcal{N}^n$ is convex if and only if $\mathcal{L}(\mathcal{F}_i) - \mathcal{M} = \emptyset$. For each member $\widetilde{\mathbf{m}}$ of $\mathcal{L}(\mathcal{F}_i)$, theorem 4.3 yields a polynomial time algorithm that decides if $\widetilde{\mathbf{m}} \in \mathcal{M}$. Now, we address the issue of estimating an upper bound on the size of $\mathcal{L}(\mathcal{F}_i)$.

Let:

$$\epsilon_i := max(\mathbf{m}_{1_i}, \mathbf{m}_{2_i}, \dots, \mathbf{m}_{k_i}) - min(\mathbf{m}_{1_i}, \mathbf{m}_{2_i}, \dots, \mathbf{m}_{k_i}),$$

and $c = \max_i(\epsilon_i)$, then it follows that $card(\mathcal{L}(\mathcal{F}_i)) \leq c^n$. The quantity c can be interpreted as a measure-of-variation among the individual components of the members of $\Upsilon(\mathcal{F}_i)$. For a fixed dimension (i.e. n is fixed), it follows that the size of $\mathcal{L}(\mathcal{F}_i)$ is polynomial in the measure-of-variation c. We argue that the set $\mathcal{L}(\mathcal{F}_i)$ can be constructed in polynomial time.

Let $\beta_i = \min(\mathbf{m}_{1_i}, \dots, \mathbf{m}_{k_i})$, and

$$\mathcal{S}(\mathcal{F}_i) := \{\mathbf{x} \in \mathcal{N}^n \mid \forall i \in \{1, \dots, n\}, \mathbf{x}_i \in [\beta_i, \beta_i + c]\} \tag{4.3}$$

$\mathcal{S}(\mathcal{F}_i)$ is the superset of $\mathcal{L}(\mathcal{F}_i)$, where $card(\mathcal{S}(\mathcal{F}_i)) = c^n$. Hence,

$$\forall \mathbf{x} \in \mathcal{S}(\mathcal{F}_i) \text{ if } \mathrm{LP}(\mathbf{x}) > 0 \Rightarrow \mathbf{x} \in \mathcal{L}(\mathcal{F}_i)$$

As a consequence of the polynomial time solvability of LPs, this condition can be checked in polynomial time.

4.2 Convexity Testing Algorithm

Algorithm 1 tests the convexity of a right-closed set $\mathcal{M} \subseteq \mathcal{N}^n$. Its correctness follows directly from the results in this paper.

Algorithm 1 Testing Convexity Algorithm

Compute $\widetilde{P}$ (cf. equation 3.2 and theorem 3.2).
for Each right-closed facet $\mathcal{F}_i$ of $\widetilde{P}$ **do**
    Compute $\Upsilon(\mathcal{F}_i) = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_k\}$ (cf. equation 4.1)
    for Each $i \in \{1, 2, \ldots, n\}$ **do**
        Compute $\epsilon_i =: max(\mathbf{m}_{1_i}, \mathbf{m}_{2_i}, \ldots, \mathbf{m}_{k_i}) - min(\mathbf{m}_{1_i}, \mathbf{m}_{2_i}, \ldots, \mathbf{m}_{k_i})$.
        $c = \max_i(\epsilon_i)$.
        Compute $\mathcal{S}(\mathcal{F}_i)$ (cf. equation 4.3)
        $\mathcal{L}(\mathcal{F}_i) = \varnothing$
        for Every element $\mathbf{x} \in \mathcal{S}(\mathcal{F}_i)$ **do**
            if $(\mathrm{LP}(\mathbf{x}) > 0) \wedge (\mathbf{x} \ngeq \widetilde{\mathbf{m}}), \forall\ \widetilde{\mathbf{m}} \in \mathcal{L}(\mathcal{F}_i)$ **then**
                $\mathcal{L}(\mathcal{F}_i) = \mathcal{L}(\mathcal{F}_i) \cup \{\mathbf{x}\}$
        for Every $\widetilde{\mathbf{m}} \in \mathcal{L}(\mathcal{F}_i)$ **do**
            if $\widetilde{m} \notin \mathcal{M}$ **then**
                Not convex; Exit.
Convex; Exit.

The first step in the algorithm 1 is computing $\widetilde{P}$. From the discussion following theorem 3.2, this can be accomplished in polynomial time. By theorem 2.1, such a procedure for m points in n-dimension, can be done deterministically in $\mathcal{O}(mlog(m) + m^{\lfloor n/2 \rfloor})$; when n is fixed, this is a polynomial time operation. It is worth mentioning that we assume $card(\min(\mathcal{M})) = m$. Next step calculates the superset $\mathcal{S}(\mathcal{F}_i)$, which can be done in $\mathcal{O}(m)$ time. For each $\mathbf{x} \in \mathcal{S}(\mathcal{F}_i)$, we need to compute $LP(\mathbf{x})$, which can be accomplished in polynomial time as well. This procedure has to be done $card(\mathcal{S}(\mathcal{F}_i)) = c^n$ times, where $c = \max_i(\epsilon_i)$ is a fixed number. Therefore, calculating $\mathcal{L}(\mathcal{F}_i)$, also can be done in polynomial time. Note that $\mathcal{L}(\mathcal{F}_i)$, contains only the minimal elements of $\mathcal{L}(\mathcal{F}_i)$, which can reduce the computational time further. The last step, verifying the membership of elements in $\mathcal{L}(\mathcal{F}_i)$ to $\mathcal{M}$ will take $\mathcal{O}(m)$ time. Since the number of facets for $\widetilde{P}$ is bounded above by $\mathcal{O}(m^{\lfloor n/2 \rfloor})$, algorithm 1 will only execute polynomial number of operations, giving us a polynomial time algorithm in m and c for testing the convexity of a right-closed set, when n is fixed.

5 Conclusion

In this paper we discussed the problem of verifying the convexity for a set of integral-vectors that is right-closed. We showed that this problem is a decidable problem, unlike the similar problem over a set of real-valued vectors. For fixed dimension, we also showed that the convexity of the right-closed integer set can be done in polynomial time. This is main contribution, additional illustrative examples can be found in reference [16].

As a future direction of research, we note that there is room to improve the efficiency of the algorithm. Convexity tests that do not rely on the computation of the convex hull could possibly yield faster algorithms. If a probabilistic solution to testing convexity is satisfactory, randomized algorithms for convexity testing can be explored. This procedure finds use in the efficient synthesis of Supervisory policies that avoid livelocks in DEDS systems [8], [4].

Competing Interests

Authors have declared that no competing interests exist.

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