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COEFFICIENT ESTIMATES OF SOME CLASSES OF RATIONAL FUNCTIONS

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Abstract. Let $\mathcal{A}$ be the class of analytic and univalent functions in the open unit disc Δ normalized such that $f(0) = 0 = f'(0) - 1$. In this paper, for $\psi \in \mathcal{A}$ of the form $\frac{z}{f(z)}$, $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $0 \leq \alpha \leq 1$, we introduce and investigate interesting subclasses $\mathcal{H}_\sigma(\phi)$, $S_\sigma(\alpha, \phi)$, $M_\sigma(\alpha, \phi)$, $\Im_\alpha(\alpha, \phi)$ and $\beta_\alpha(\lambda, \phi)\,(\lambda \geq 0)$ of analytic and bi-univalent Ma-Minda starlike and convex functions. Furthermore, we find estimates on the coefficients $|a_1|$ and $|a_2|$ for functions in these classess. Several related classes of functions are also considered.

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1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Also, by $\wp$ we shall denote the subclass of all functions in $\mathcal{A}$ which are univalent in Δ. Let P denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

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which are analytic in Δ such that

$$p(0) = 1 \text{ and } \mathrm{Re}\{p(z)\} > 0 \quad (z \in \Delta).$$

If the functions f and g are analytic in Δ, then f is said to be subordinate to g, written $f(z) \prec g(z)$, provided there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ so that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\triangle$ then we have the following equivalence (see for details, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\triangle) \subset g(\triangle).$$

Some of the important and well-investigated subclasses of the univalent function class $\wp$ include (for example) the class $S(\alpha)$ of starlike functions of order α in Δ and the class $C(\alpha)$ of convex functions of order α in Δ. By definition, we have

$$S(\alpha) = \left\{ f : f \in \wp \text{ and } \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha \quad (z \in \Delta,\ 0 \le \alpha < 1) \right\} \tag{1}$$

and

$$C(\alpha) = \left\{ f : f \in \wp \text{ and } \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \Delta,\ 0 \le \alpha < 1) \right\}. \tag{2}$$

It readily follows from the definitions (1) and (2) that

$$f(z) \in C(\alpha) \iff zf'(z) \in S(\alpha). \tag{3}$$

It is well known that for each $f \in \wp$, the koebe one-quarter theorem [13] ensures the image of Δ under f contains a disk of radius 1/4. Thus every univalent function $f \in \wp$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \ (|z| < 1)$$

and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f),\ r_0(f) \ge 1/4).$$

In fact, the inverse function $g = f^{-1}$ is defined by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2a_3 + a_4)w^4 + ...$$

A function $f \in \mathcal{A}$ is said to bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let σ denote the class of bi-univalent functions defined in the unit disk Δ and let $\phi \in P$ and $\phi(\Delta)$ is symmetric with respect to the the real axis, such a function has a Taylor series of the form:

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ... \, (B_1 > 0). \tag{4}$$

In [14], the authors introduced the class $S(\phi)$ of the so-called Ma and Minda starlike functions and the class $C(\phi)$ of Ma and Minda convex functions, unifying several previously studied classes related to those of starlike and convex functions. The class $S(\phi)$ consists of all the functions $f \in \mathcal{A}$ satisfying subordination $\dfrac{zf'(z)}{f(z)} \prec \phi(z)$, whereas $C(\phi)$ is formed with functions $f \in \mathcal{A}$ for which the subordination $1+\dfrac{zf''(z)}{f'(z)} \prec \phi(z)$ holds. Lewin [15] investigated the class σ and showed that $|a_2| < 1.51$ for function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \sigma$. Subsequently, Brannan and Clunie [16] conjectured that $|a_2| < \sqrt{2}$. Netanyahu [17], on the other hand, showed that max $|a_2| = 4/3$ if $f(z) \in \sigma$. Brannan and Taha [18] and Taha [19] introduced certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike and convex functions, they introduced bi-starlike functions and bi-convex functions and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Recently, many authors investigated bounds for various subclasses of bi-univalent functions (see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. In [34], Mitrinovic essentially investigated certain geometric properties of functions ψ of the form

$$\psi(z) = \frac{z}{f(z)}, \quad f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n. \tag{5}$$

In [35], Reade et al. derived coefficient conditions that guarantee the univalence, starlikeness or convexity of rational functions of the form (5), these results have been improved and generalized in [36]. In this paper, estimates on the initial coefficients for bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type of rational form (5) are obtained. Several related classes are also considered.
In order to derive our main results, we require the following lemma.

Lemma 1.1. *(see [37]) If $p(z) \in P$, then*

$$|c_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, ...\}). \tag{6}$$

2. Coefficients estimates

A function $\psi(z) \in \mathcal{A}$ with Re $(\psi'(z)) > 0$ is known to be univalent. This motivates the following class of functions.

Definition 2.1. A function $\psi \in \sigma$given by (5) is said to be in the class $\mathcal{H}_\sigma(\phi)$ if the following conditions are satisfied:

$$\psi'(z) \prec \phi(z)\,(z \in \Delta) \quad \text{and} \quad g'(w) \prec \phi(w)\,(w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + ...\,(0 < \gamma \leq 1,\ z \in \Delta)$$

in Definition 2.1 of the bi-univalent function class $\mathcal{H}_\sigma(\phi)$ we obtain a new class $\mathcal{H}_\sigma(\gamma)$ given by Definition 2.2 below.

Definition 2.2. For $0 < \gamma \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $\mathcal{H}_\sigma(\gamma)$ if the following conditions are satisfied:

$$\psi'(z) \prec \left(\frac{1+z}{1-z}\right)^\gamma (z \in \Delta) \quad \text{and} \quad g'(w) \prec \left(\frac{1+w}{1-w}\right)^\gamma (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1 + 2(1-\nu)z + 2(1-\nu)z^2 + ... (0 < \nu \leq 1,\ z \in \Delta)$$

in Definition 2.1 of the bi-univalent function class $\mathcal{H}_\sigma(\phi)$ we obtain, a new class $\mathcal{H}_\sigma(\nu)$ given by Definition 2.3 below.

Definition 2.3. For $0 < \nu \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $\mathcal{H}_\sigma(\nu)$ if the following conditions hold true:

$$\psi'(z) \prec \frac{1+(1-2\nu)z}{1-z} (z \in \Delta) \quad \text{and} \quad g'(w) \prec \frac{1+(1-2\nu)w}{1-w} (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

Theorem 2.4. *Let $\psi(z) \in \mathcal{H}_\sigma(\phi)$ be of the form (5). Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|3B_1^2 - 4B_2 + 4B_1|}} \quad \textit{and } |a_2| \leq \frac{1}{3}B_1. \tag{7}$$

Proof. Let $\psi(z) \in \mathcal{H}_\sigma(\phi)$ and $g = \psi^{-1}$. Then there exist two functions u and v, analytic in Δ, with $u(0) = v(0) = 0$, $|u(z)| < 1$ and $|v(w)| < 1$, $z, w \in \Delta$, such that

$$\psi'(z) = \phi(u(z)) \quad \text{and} \quad g'(w) = \phi(v(w)). \tag{8}$$

Next, define the functions p_1 and p_2 by

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1+c_1z+c_2z^2+... \text{ and } \quad p_2(w) = \frac{1+v(w)}{1-v(w)} = 1+b_1w+b_2^2w^2+...,$$

or, equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + ...\right], \tag{9}$$

and

$$v(w) = \frac{p_2(w)-1}{p_2(w)+1} = \frac{1}{2}\left[b_1w + \left(b_2 - \frac{b_1^2}{2}\right)w^2 + ...\right]. \tag{10}$$

Then p_1 and p_2 analytic in Δ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \Delta \longrightarrow \Delta$, the functions p_1 and p_2 have a positive real part in Δ, and $|b_i| \leq 2$ and $|c_i| \leq 2$. Clearly, upon substituting from (9) and (10) into (8), if we make use of (4), we find that

$$\psi'(z) = \phi(\frac{p_1(z)-1}{p_1(z)+1}) = 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]z^2 + ..., \tag{11}$$

and

$$g'(w) = \phi(\frac{p_2(w)-1}{p_2(w)+1}) = 1 + \frac{1}{2}B_1 b_1 w + \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right] w^2 + ... \tag{12}$$

Since $\psi \in \sigma$ has the Maclaurin's series given by

$$\psi(z) = z - a_1 z^2 + (a_1^2 - a_2)z^3 + ..., \tag{13}$$

a computation shows that its inverse $g = \psi^{-1}$ has the expansion

$$g(w) = \psi^{-1}(w) = w + a_1 w^2 + (a_1^2 + a_2)w^3 + ... \tag{14}$$

Using (13) and (14) in (11) and (12) respectively, we get

$$-2a_1 = \frac{1}{2}B_1 c_1 \tag{15}$$

$$3(a_1^2 - a_2) = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2 c_1^2, \tag{16}$$

$$2a_1 = \frac{1}{2}B_1 b_1 \tag{17}$$

and

$$3(a_1^2 + a_2) = \frac{1}{2}B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{4}B_2 b_1^2. \tag{18}$$

From (15) and (17), we have

$$c_1 = -b_1. \tag{19}$$

Adding (16) and (18) and then using (15) and (19), we get

$$a_1^2 = \frac{B_1^3(c_2 + b_2)}{4(3B_1^2 - 4B_2 + 4B_1)},$$

and now, by applying Lemma 1.1 for the coefficients b_2 and c_2, the last equation gives the bound of $|a_1|$ from (7). By subtracting (18) from (16), further computations using (19) lead to

$$a_2 = \frac{1}{12}B_1(b_2 - c_2).$$

The bound of $|a_2|$, as asserted in (7), is now a consequence of Lemma 1.1, and this completes our proof. □

Using the parameter setting of Definition 2.2 in Theorem 2.4, we get the following corollary.

Corollary 2.5. *For* $0 < \gamma \leq 1$, *let the function* $\psi \in \mathcal{H}_\sigma(\gamma)$ *be of the form (5). Then*

$$|a_1| \leq \frac{\sqrt{2}\gamma}{\sqrt{\gamma + 2}} \quad \text{and} \quad |a_2| \leq \frac{2}{3}\gamma.$$

Using the parameter setting of Definition 2.3 in Theorem 2.4, we get the following corollary.

Corollary 2.6. *For $0 < \nu \leq 1$, let the function $\psi \in \mathcal{H}_\sigma(\nu)$ be given by (5). Then*

$$|a_1| \leq \sqrt{\frac{2}{3}(1-\nu)} \quad \textit{and} \ |a_2| \leq \frac{2}{3}(1-\nu).$$

Definition 2.7. A function $\psi \in \sigma$ is given by (5) is said to be in the class $S_\sigma(\alpha, \phi)$ if the following subordinations hold:

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2 \psi''(z)}{\psi(z)} \prec \phi(z)\,(z \in \Delta) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} \prec \phi(w)\,(w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + ...\,(0 < \gamma \leq 1,\ z \in \Delta)$$

in Definition 2.7 of the bi-univalent function class $S_\sigma(\alpha, \phi)$, we obtain a new class $S_\sigma(\alpha, \gamma)$ given by Definition 2.8 below.

Definition 2.8. For $0 \leq \alpha \leq 1$ and $0 < \gamma \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $S_\sigma(\alpha, \gamma)$ if the following subordinations hold:

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2 \psi''(z)}{\psi(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma (z \in \Delta),$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} \prec \left(\frac{1+w}{1-w}\right)^\gamma (w \in \Delta),$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1 + 2(1-\nu)z + 2(1-\nu)z^2 + ...\,(0 < \nu \leq 1,\ z \in \Delta)$$

in Definition 2.7 of the bi-univalent function class $S_\sigma(\alpha, \phi)$ we obtain a new class $S_\sigma(\alpha, \nu)$ given by Definition 2.9 below.

Definition 2.9. For $0 \leq \alpha \leq 1$ and $0 < \nu \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $S_\sigma(\alpha, \nu)$ if the following subordinations hold:

$$\frac{z\psi'(z)}{\psi(z)} + \frac{\alpha z^2 \psi''(z)}{\psi(z)} \prec \frac{1+(1-2\nu)z}{1-z}\,(z \in \Delta)$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} \prec \frac{1+(1-2\nu)w}{1-w}\,(w \in \Delta),$$

where $g(w) = \psi^{-1}(w)$.

Note that $S(\phi) = S_\sigma(0, \phi)$. For functions in the class $S_\sigma(\alpha, \phi)$, the following coefficient estimates are obtained,

Theorem 2.10. *Let $\psi(z) \in S_\sigma(\alpha, \phi)$ be of the form (5). Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2(1+4\alpha)+(B_1-B_2)(1+2\alpha)^2|}}, \tag{20}$$

and

$$|a_2| \leq \frac{B_1}{1+3\alpha}. \tag{21}$$

Proof. Let $\psi \in S_\sigma(\alpha, \phi)$, there are two Schwarz functions u and v defined by (9) and (10) respectively, such that

$$\frac{z\psi'(z)}{\psi(z)}+\frac{\alpha z^2\psi''(z)}{\psi(z)} = \phi(u(z)) \text{ and } \frac{wg'(w)}{g(w)}+\frac{\alpha w^2 g''(w)}{g(w)} = \phi(v(w)), \quad \left(g=\psi^{-1}\right). \tag{22}$$

Since

$$\frac{z\psi'(z)}{\psi(z)}+\frac{\alpha z^2\psi''(z)}{\psi(z)} = 1-(1+2\alpha)\,a_1 z+\left[(1+4\alpha)\,a_1^2-2\,(1+3\alpha)\,a_2\right]z^2+...$$

and

$$\frac{wg'(w)}{g(w)}+\frac{\alpha w^2 g''(w)}{g(w)} = 1+(1+2\alpha)\,a_1 w+\left[(1+4\alpha)\,a_1^2+2\,(1+3\alpha)\,a_2\right]w^2+...,$$

then (11), (12) and (22) yields

$$-(1+2\alpha)a_1 = \frac{1}{2}B_1 c_1 \tag{23}$$

$$(1+4\alpha)a_1^2-2(1+3\alpha)a_2 = \frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2 c_1^2, \tag{24}$$

$$(1+2\alpha)a_1 = \frac{1}{2}B_1 b_1 \tag{25}$$

and

$$(1+4\alpha)a_1^2+2(1+3\alpha)a_2 = \frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2 b_1^2. \tag{26}$$

From (23) and (25), we get

$$c_1 = -b_1, \tag{27}$$

and after some further calculations using (24)-(27) we find

$$a_1^2 = \frac{B_1^3(c_2+b_2)}{4\left[B_1^2(1+4\alpha)+(B_1-B_2)(1+2\alpha)^2\right]},$$

and

$$a_2 = \frac{B_1(b_2-c_2)}{4(1+3\alpha)}.$$

Applying Lemma 1.1, the estimates in (20) and (21) follow. □

For $\alpha = 0$, Theorem 2.10 readily yields the following coefficient estimates for Ma-Minda bi-starlike functions.

Corollary 2.11. *Let ψ given by (5) be in the class $S(\phi)$. Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}}, \quad \text{and} \quad |a_2| \leq B_1.$$

Using the parameter setting of Definition 2.8 in Theorem 2.10, we get the following corollary.

Corollary 2.12. *For $0 \leq \alpha \leq 1$ and $0 < \gamma \leq 1$, let the function $\psi \in S_\sigma(\alpha, \gamma)$ be of the form (5). Then*

$$|a_1| \leq \frac{2\gamma}{\sqrt{(1+2\alpha)^2 + \gamma\left[1 + 4\alpha - 4\alpha^2\right]}} \quad \text{and} \quad |a_2| \leq \frac{2\gamma}{1+3\alpha}.$$

Using the parameter setting of Definition 2.9 in Theorem 2.10 we get the following corollary.

Corollary 2.13. *For $0 \leq \alpha \leq 1$ and $0 < \nu \leq 1$, let the function $\psi \in S_\sigma(\alpha, \nu)$ be of the form (5). Then*

$$|a_1| \leq \sqrt{\frac{2\,(1-\nu)}{1+4\alpha}} \quad \text{and} \quad |a_2| \leq \frac{2\,(1-\nu)}{1+3\alpha}.$$

Definition 2.14. A function $\psi \in \sigma$ given by (5) belongs to the class $M_\sigma(\alpha, \phi)$ $(0 \leq \alpha \leq 1)$, if the following subordinations hold:

$$(1-\alpha)\frac{z\psi'(z)}{\psi(z)} + \alpha(1 + \frac{z\psi''(z)}{\psi'(z)}) \prec \phi(z)\,(z \in \Delta)\,,$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha(1 + \frac{wg''(w)}{g'(w)}) \prec \phi(w), (w \in \Delta)\,,$$

where $g(w) := \psi^{-1}(w)$.

If we set

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \ldots\,(0 < \gamma \leq 1,\ z \in \Delta)$$

in Definition 2.14 of the bi-univalent function class $M_\sigma(\alpha, \phi)$, we obtain a new class $M_\sigma(\alpha, \gamma)$ given by Definition 2.15 below.

Definition 2.15. For $0 \leq \alpha \leq 1$ and $0 < \gamma \leq 1$, a function $\psi \in \sigma$ given by (5) is said to be in the class $M_\sigma(\alpha, \gamma)$ if the following subordinations hold:

$$(1-\alpha)\frac{z\psi'(z)}{\psi(z)} + \alpha(1 + \frac{z\psi''(z)}{\psi'(z)}) \prec \left(\frac{1+z}{1-z}\right)^\gamma (z \in \Delta)\,,$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha(1 + \frac{wg''(w)}{g'(w)}) \prec \left(\frac{1+w}{1-w}\right)^\gamma (w \in \Delta)\,,$$

$g(w) := \psi^{-1}(w)$.

Corollary 2.16. *If we set*

$$\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1+2(1-\nu)z+2(1-\nu)z^2+...\,(0<\nu\leq 1,\ z\in\Delta)$$

in Definition 2.14 of the bi-univalent function class $M_\sigma(\alpha,\phi)$ we obtain a new class $M_\sigma(\alpha,\nu)$ given by Definition 2.17 below.

Definition 2.17. For $0\leq\alpha\leq 1$ and $0<\nu\leq 1$, a function $\psi\in\sigma$ given by (5) is said to be in the class $M_\sigma(\alpha,\nu)$ if the following subordinations hold:

$$(1-\alpha)\frac{z\psi'(z)}{\psi(z)}+\alpha(1+\frac{z\psi''(z)}{\psi'(z)})\prec\frac{1+(1-2\nu)z}{1-z}\,(z\in\Delta)\,,$$

and

$$(1-\alpha)\frac{w\psi'(w)}{\psi(w)}+\alpha(1+\frac{w\psi''(w)}{\psi'(w)})\prec\frac{1+(1-2\nu)w}{1-w}\,(w\in\Delta)\,,$$

where $g(w):=\psi^{-1}(w)$.

A function in the class $M_\sigma(\alpha,\phi)$ is called bi-Mocanu-convex function of Ma-Minda type. This class unifies the classes $S(\alpha)$ and $C(\alpha)$. For functions in the class $M_\sigma(\alpha,\phi)$, the following coefficients estimates hold.

Theorem 2.18. *Let $\psi(z)\in M_\sigma(\alpha,\phi)$ be of the form (5). Then*

$$|a_1|\leq\frac{B_1\sqrt{B_1}}{\sqrt{(1+\alpha)\left|B_1^2+(1+\alpha)(B_1-B_2)\right|}},\tag{28}$$

and

$$|a_2|\leq\frac{B_1}{2(1+2\alpha)}.\tag{29}$$

Proof. If $\psi\in M_\sigma(\alpha,\phi)$, then there exist are two Schwarz functions u and v defined by (9) and (10) respectively, such that

$$(1-\alpha)\frac{z\psi'(z)}{\psi(z)}+\alpha(1+\frac{z\psi''(z)}{\psi'(z)})=\phi(u(z)),\tag{30}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha(1+\frac{wg''(w)}{g'(w)})=\phi(v(w)).\tag{31}$$

Since

$$(1-\alpha)\frac{z\psi'(z)}{\psi(z)}+\alpha(1+\frac{z\psi''(z)}{\psi'(z)})=1-(1+\alpha)\,a_1z+\left[(1+\alpha)\,a_1^2-2\,(1+2\alpha)\,a_2\right]z^2+...$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha(1+\frac{wg''(w)}{g'(w)})=1+(1+\alpha)\,a_1w+\left[(1+\alpha)\,a_1^2+2\,(1+2\alpha)\,a_2\right]w^2+...,$$

from (11), (12), (30) and (31), it follows that

$$-(1+\alpha)a_1=\frac{1}{2}B_1c_1,\tag{32}$$

$$(1+\alpha)a_1^2 - 2(1+2\alpha)a_2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2, \tag{33}$$

$$(1+\alpha)a_1 = \frac{1}{2}B_1b_1, \tag{34}$$

and

$$(1+\alpha)a_1^2 + 2(1+2\alpha)a_2 = \frac{1}{2}B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{4}B_2b_1^2, \tag{35}$$

Eqs. (32) and (34) yields

$$c_1 = -b_1, \tag{36}$$

and after some further calculations using (33)-(35) we find

$$a_1^2 = \frac{B_1^3(c_2+b_2)}{4(1+\alpha)\left[B_1^2 + (1+\alpha)(B_1 - B_2)\right]},$$

and

$$a_2 = \frac{B_1\,(b_2 - c_2)}{8(1+2\alpha)},$$

Applying Lemma 1.1, the estimates in (28) and (29) follow. □

For $\alpha = 0$, Theorem 2.18 gives the coefficient estimates for Ma-Minda bi-starlike functions, while for $\alpha = 1$, it gives the following estimates for Ma-Minda bi-convex functions.

Corollary 2.19. *Let ψ given by (5) be in the class $C(\phi)$. Then*

$$|a_1| \le \frac{B_1\sqrt{B_1}}{2\left|B_1^2 + 2(B_1 - B_2)\right|}, \quad \textit{and} \quad |a_2| \le \frac{B_1}{6}.$$

Using the parameter setting of Definition 2.15 in Theorem 2.18 we get the following corollary.

Corollary 2.20. *For $0 \le \alpha \le 1$ and $0 < \gamma \le 1$, let the function $\psi \in M_\sigma(\alpha,\gamma)$ be of the form (5). Then*

$$|a_1| \le \frac{2\gamma}{\sqrt{(1+\alpha)\left[(1+\alpha) + \gamma\,(1-\alpha)\right]}} \quad \textit{and} \quad |a_2| \le \frac{\gamma}{1+2\alpha}.$$

Using the parameter setting of Definition 2.17 in Theorem 2.18 we get the following corollary.

Corollary 2.21. *For $0 \le \alpha \le 1$ and $0 < \nu \le 1$, let the function $\psi \in M_\sigma(\alpha,\nu)$ be of the form (5). Then*

$$|a_1| \le \sqrt{\frac{2\,(1-\nu)}{1+\alpha}} \quad \textit{and} \;\; |a_2| \le \frac{(1-\nu)}{1+2\alpha}.$$

Definition 2.22. A function $\psi \in \sigma$ given by (5) is said to be in the class $\Im_\alpha(\alpha, \phi)\,(0 \le \alpha \le 1)$, if the following subordinations hold:

$$\left(\frac{z\psi'(z)}{\psi(z)}\right)^\alpha \left(1 + \frac{z\psi''(z)}{\psi'(z)}\right)^{1-\alpha} \prec \phi(z)\,(z \in \Delta)\,,$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\alpha \left(1 + \frac{wg''(w}{g'(w)}\right)^{1-\alpha} \prec \phi(w)\,(w \in \Delta)\,,$$

$g(w) := \psi^{-1}(w)$. This class also reduces to classes of Ma-Minda bi-starlike and bi-convex functions. For functions in this class, the following coefficient estimates are obtained.

Theorem 2.23. *Let $\psi(z) \in \Im_\alpha(\alpha, \phi)$ be of the form (5). Then*

$$|a_1| \le \frac{2B_1\sqrt{B_1}}{\sqrt{|2\,(\alpha^2 - 3\alpha + 4)\,B_1^2 + 4(\alpha-2)^2(B_1 - B_2)|}}, \tag{37}$$

and

$$|a_2| \le \frac{B_1}{2\,|3 - 2\alpha|}. \tag{38}$$

Proof. Let $\psi \in \Im_\alpha(\alpha, \phi)$, then there exist are two Schwarz functions u and v defined by (9) and (10) respectively, such that

$$\left(\frac{z\psi'(z)}{\psi(z)}\right)^\alpha \left(1 + \frac{z\psi''(z)}{\psi'(z)}\right)^{1-\alpha} = \phi(u(z)) \tag{39}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^\alpha \left(1 + \frac{wg''(w}{g'(w)}\right)^{1-\alpha} = \phi(v(w)). \tag{40}$$

Since

$$\left(\frac{z\psi'(z)}{\psi(z)}\right)^\alpha \left(1 + \frac{z\psi''(z)}{\psi'(z)}\right)^{1-\alpha} = 1 - (2-\alpha)\,a_1 z + \left[\frac{\alpha^2 - 3\alpha + 4}{2}a_1^2 - 2\,(3 - 2\alpha)\,a_2\right] z^2 + \ldots\ .$$

Also

$$\left(\frac{wg'(w)}{g(w)}\right)^\alpha \left(1 + \frac{wg''(w}{g'(w)}\right)^{1-\alpha} = 1 + (2-\alpha)\,a_1 w + \left[\frac{\alpha^2 - 3\alpha + 4}{2}a_1^2 + 2\,(3 - 2\alpha)\,a_2\right] w^2 + \ldots,$$

from (11), (12), (39) and (40), it follows that

$$-\,(2-\alpha)a_1 = \frac{1}{2}B_1 c_1, \tag{41}$$

$$\frac{\alpha^2 - 3\alpha + 4}{2}a_1^2 - 2\,(3 - 2\alpha)\,a_2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2 c_1^2, \tag{42}$$

$$(2-\alpha)a_1 = \frac{1}{2}B_1 b_1 \tag{43}$$

and

$$\frac{\alpha^2-3\alpha+4}{2}a_1^2 + 2\,(3-2\alpha)\,a_2 = \frac{1}{2}B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{4}B_2 b_1^2. \tag{44}$$

Eqs. (41) and (43) obviously yield

$$c_1 = -b_1. \tag{45}$$

Eqs. (42)-(44) and (45) lead to

$$a_1^2 = \frac{B_1^3(c_2+b_2)}{2\,(\alpha^2-3\alpha+4)\,B_1^2 + 4(\alpha-2)^2(B_1-B_2)}.$$

By applying Lemma 1.1, we get the desired estimate of $|a_1|$ as asserted in (37). Proceeding similarly as in the earlier proof, using (42)-(45), it follows that

$$a_2 = \frac{B_1(b_2-c_2)}{8(3-2\alpha)},$$

which, in view of Lemma 1.1, yields the estimate (38). □

Definition 2.24. A function $\psi \in \sigma$ given by (5) is said to be in the class $\beta_\alpha(\lambda,\phi),\ \lambda \geq 0$, if the following subordinations hold:

$$(1-\lambda)\,\frac{\psi(z)}{z} + \lambda\psi'(z) \prec \phi(z)\,(z\in\Delta)\,,$$

and

$$(1-\lambda)\,\frac{g(w)}{w} + \lambda g'(w) \prec \phi(w)\,(w\in\Delta)\,,$$

where $g(w) := \psi^{-1}(w)$.

Theorem 2.25. *Let $\psi(z) \in \beta_\alpha(\lambda,\phi),\ \lambda \geq 0$ be of the form (5). Then*

$$|a_1| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|(1+2\lambda)\,B_1^2 + (1+\lambda)^2(B_1-B_2)|}}, \tag{46}$$

and

$$|a_2| \leq \frac{B_1}{1+2\lambda}. \tag{47}$$

Proof. Let $\psi \in \beta_\alpha(\lambda,\phi)$, then there exist are two Schwarz functions u and v defined by (9) and (10) respectively, such that

$$(1-\lambda)\,\frac{\psi(z)}{z} + \lambda\psi'(z) = \phi(u(z)) \tag{48}$$

and

$$(1-\lambda)\,\frac{g(w)}{w} + \lambda g'(w) = \phi(v(w)). \tag{49}$$

Since

$$(1-\lambda)\,\frac{\psi(z)}{z} + \lambda\psi'(z) = 1-(1+\lambda)\,a_1 z + \left[(1+2\lambda)\left(a_1^2-a_2\right)\right]z^2 + ...,$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)=1+(1+\lambda)\,a_1 w+\left[(1+2\lambda)\left(a_1^2+a_2\right)\right]w^2+...,$$

from (11), (12), (48) and (49), it follows that

$$-(1+\lambda)a_1=\frac{1}{2}B_1c_1, \tag{50}$$

$$(1+2\lambda)(a_1^2-a_2)=\frac{1}{2}B_1(c_2-\frac{c_1^2}{2})+\frac{1}{4}B_2c_1^2, \tag{51}$$

$$(1+\lambda)a_1=\frac{1}{2}B_1b_1 \tag{52}$$

and

$$(1+2\lambda)(a_1^2+a_2)=\frac{1}{2}B_1(b_2-\frac{b_1^2}{2})+\frac{1}{4}B_2b_1^2. \tag{53}$$

Now (50) and (52) clearly yield

$$c_1=-b_1. \tag{54}$$

Eqs. (51), (53) and (54) lead to

$$a_1^2=\frac{B_1^3(c_2+b_2)}{4\left[(1+2\lambda)\,B_1^2+(1+\lambda)^2\,(B_1-B_2)\right]},$$

By applying Lemma 1.1, we get the desired estimate of $|a_1|$ as asserted in (46). Proceeding similarly as in the earlier proof, using (51)-(54), it follows that

$$a_2=\frac{B_1(b_2-c_2)}{4(1+2\lambda)},$$

which, in view of Lemma 1.1, yields the estimate (47). □

Competing Interests

The authors declare that they have no competing interests.

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